THE PRINCIPLES

OF

GRAPHIC STATICS

BY

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PREFACE TO SECOND EDITION.

SINCE this book was published, nearly nine years have elapsed, during which the subject has made steady progress in this country. Systematic teaching has to a great extent taken the place of the desultory adoption of a method here and there in a course of engineering study. Graphic Statics tends more and more to be read on the ground of its educational value, and not merely for what it can accomplish; while it is now generally recognised that no real grasp of the subject, no certainty of handling, can be attained without a mastery of principles.

Under these circumstances, and to meet the demand which exists both here and in America, it has been decided to issue a cheaper edition. Pressure of other work has rendered it impossible for the writer to make additions which would perhaps add to such usefulness as the book may claim. He has been obliged, therefore, to content himself with revision and the minor amplifications suggested by small difficulties which have been experienced by various correspondents. The proofs have been read by Mr. T. H. Eagles, M.A., Instructor in Geometrical Drawing and Lecturer in Architecture at the R. I. E. College, Coopers Hill, to whose kind assistance, generously proffered, the writer is deeply indebted.

LONDON,
May, 1888.

PREFACE.

THE study of Graphic Statics, as a subject *sui generis*, has made but little progress in England, though the great value of numerous Graphic methods has long been fully acknowledged. While in many of the great Engineering Schools of the Continent the subject is deemed worthy of a professorial chair, in England it is left to be gleaned almost hap-hazard. A method more or less is thrown into a course of teaching, according to the predilection or dislike of the teacher for Graphic modes of procedure.

And yet the subject is valuable, not merely as a means to an end, but as a part of mental training. A mind brought up only on mathematical symbols is but half trained: the Graphic is the complement of the Analytical process. And the power conferred by Graphic method is to a large extent at the disposal of those who have had but little mathematical training. The writer once had occasion to explain a practical application of the triangle of forces to a class of working men, who seemed at once to grasp and appreciate it.

The present work is an attempt to steer a middle course between the too pronouncedly abstract character of many of the numerous foreign treatises, and the too narrowly practical treatment the subject has received in England. Though confessedly incomplete, it will, it is hoped, prove suggestive, and may perhaps lead some of its readers to a further prosecution of a really fascinating study—one, too, in which very much still remains to be done.

In order to make the work self-contained, an Appendix is added, giving tables of the weights and strength of materials. By the help of these tables it will be found possible to apply the various constructions to actual practice.

The scale to which the figures are drawn is necessarily contracted, but it has been endeavoured to make these figures clear and easy to follow.

It is perhaps necessary to apologize for the employment of a few words not yet quite naturalized in the sense here given to them. Thus "pressure" has been adopted for compressive stress, which has usually been termed "compression." Using "pressure" in this sense, "compression" is available for the alteration of length due to "pressure." Tension and extension are universally so used, and analogy seems to demand pressure and compression. The words "vector" and "stress centre" are also adopted, while "kern" has been borrowed bodily from the German. The latter word has no English equivalent which possesses a sufficiently suggestive meaning, and the thing meant—the locus of the stress centre under certain conditions—has hitherto received no English name.

Besides the ordinary English text books, the following works have been put in requisition, viz.:—

'Wrought-iron Bridges and Roofs.'—Unwin.
'La Statique Graphique.'—Levy.
'Der Constructeur.'—Reuleaux.
'Die Graphische Statik.'—Culmann.
'Die Grundzüge des Graphischen Rechnens.'—Von Ott.
'Elemente der Graphischen Statik.'—Bauschinger.
'Taschenbuch der Hütte.'

The writer is, moreover, much indebted to his colleague, Professor W. C. Unwin, B.Sc., for many valuable hints, and to his late and present colleagues, Professor A. G. Greenhill, M.A., and Mr. T. H. Eagles, M.A., for kindly reading proofs at different times.

COOPERS HILL,
November, 1879.

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GRAPHIC STATICS.

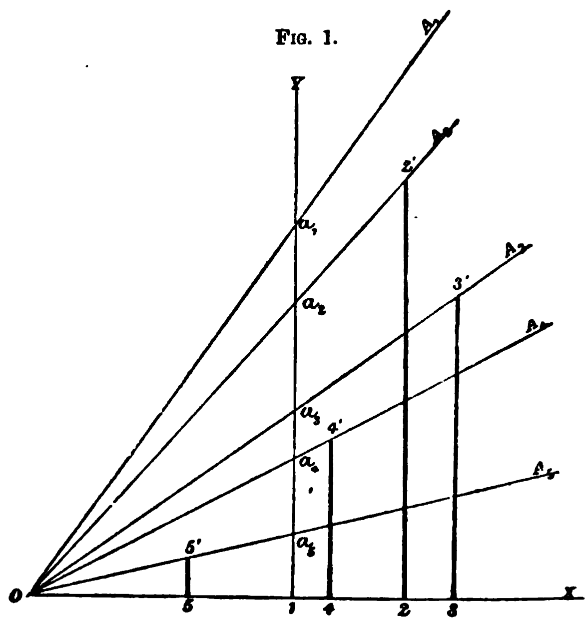
CHAPTER I.

GRAPHIC ARITHMETIC.

By Graphic Arithmetic is meant the employment of lines to represent numbers, and the performance, by means of construction, of the ordinary processes of arithmetic. The results obtained are lines which represent the required product, quotient, &c., on the same scale as that of the lines dealt with.

Thus by the square root of a given line is meant a second line which represents the square root of the number which the first line is taken to mean, and which represents that square root *on the same scale* as that on which the given line represents the number. It will be seen, therefore, that if a line is said to represent a number, a scale is always implied. In dealing with all questions, the scale must never be lost sight of.

Fig. 1.



Although in the following pages *lines* only are spoken of, it must be understood that *numbers represented by lines on some convenient scale* are meant, and by *unity* is implied a *unit length* on this scale.

1. Multiplication of Lines.—To determine the product of any number of lines

$$a_1 \times a_2 \times a_3 \times \ldots \times a_n$$

draw any line O X (Fig. 1), and set off from O a length O 1 equal to unity. At 1 erect a perpendicular O Y, and set off from 1 the distances $1\,a_1 = a_1$, $1\,a_2 = a_2$, and so on; join the points $a_1, a_2 \ldots$ to O, and produce the lines $O\,a_1$, $O\,a_2 \ldots$ indefinitely.

From O set off $O\,2 = a_1$ along O X, and raise a perpendicular $2\,2'$ at 2, cutting $O\,A_2$ in $2'$.

Then
$$\frac{22'}{O\,2} = \frac{1\,a_2}{O\,1}; \text{ or, } 22' = a_1 \times a_2.$$

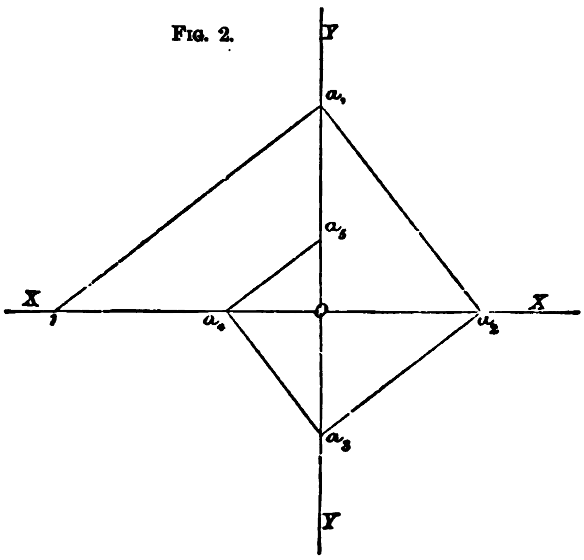
From O set off O 3 = 2 2′ along O X, and at 3 erect a perpendicular 3 3′, cutting O A_3 in 3′.

Then
$$\frac{33'}{O\,3} = \frac{1\,a_3}{O\,1}; \text{ or, } 33' = 22' \times a_3 = a_1 \times a_2 \times a_3.$$

Proceeding thus, the last ordinate 5 5′ gives the product $a_1 \times a_2 \times a_3 \times a_4 \times a_5$, which must of course be read off on the same scale as that used for the lines and for the unity length O 1.

2. Raising a Line to any Power.—Draw two axes X X, Y Y (Fig. 2), at right angles to each other, intersecting in O. From O set off O 1 along O X equal to unity and O a_1 along O Y equal to a, the line to be raised to any power, say 5.

Fig. 2.



Draw $a_1 a_2$ at right angles to 1 a_1, cutting X X in a_2; $a_2 a_3$ at right angles to $a_1 a_2$, cutting Y Y in a_3; and so on, cutting each axis in turn, and drawing each line at right angles to the last.

Then
$$\frac{O\,a_2}{O\,a_1} = \frac{O\,a_1}{O\,1}; \text{ or, } O\,a_2 = a^2.$$

Similarly,
$$\frac{O\,a_3}{O\,a_2} = \frac{O\,a_1}{O\,1}; \text{ or, } O\,a_3 = O\,a_2 \times a = a^3.$$

And finally,
$$O\,a_5 = a^5.$$

3. Extraction of Roots.—The value of $\sqrt{a}$ can be obtained by determining a mean proportion (x) between a and unity; thus $x = \sqrt{a \times 1} = \sqrt{a}$. By repeating this process the values of $\sqrt[4]{a}$, $\sqrt[8]{a}$, $\sqrt[16]{a}$, &c., can be determined; other roots can, however, be obtained by means of the logarithmic spiral or the logarithmic curve.

To draw the former, make O a (Fig. 4) equal to unity, and taking O as pole, draw any number of radii, $r_1, r_2, r_3 \ldots$ at equal angular distances, a. Now the required curve cuts every radius vector at the same angle. Hence, if 1, 2, 3 are the points in which the curve cuts r_1, r_2, r_3, the triangles 3 O 2, 2 O 1, 1 O a, &c., will, if the angles a are supposed to diminish indefinitely, be similar. Therefore

$$\frac{r_1}{1} = \frac{r_2}{r_1} = \frac{r_3}{r_2} = \frac{r_4}{r_3} = \ldots \frac{r_n}{r_{n-1}};$$

or
$$r_2 = r_1^2;\ r_3 = r_1^3;\ r_4 = r_1^4 \ldots r_n = r_1^n.$$

Hence successive radii at equal angular distances form a series in geometric progression of which r_1 is the ratio. This enables the curve to be drawn for any assumed value of this ratio r_1. With O (Fig. 3) as centre describe an arc $a\,a'$ with radius unity; on this arc set off a chord $a\,a'$ equal to the assumed value of r_1, produce O a, O a', describe an arc $b\,b'$ with radius equal to r_1; with O as centre and the chord r_2 of the arc $b\,b'$ as radius, describe an arc $c\,c'$; proceeding in this way the successively obtained chords r_2, r_3.... form the required geometric series. Set off the lengths r_1, r_2, r_3...., &c. (Fig. 3), along the successive radii in Fig. 4; thus the points 1, 2, 3 .... are obtained, and the curve drawn through them is part of the required logarithmic

FIG. 3.

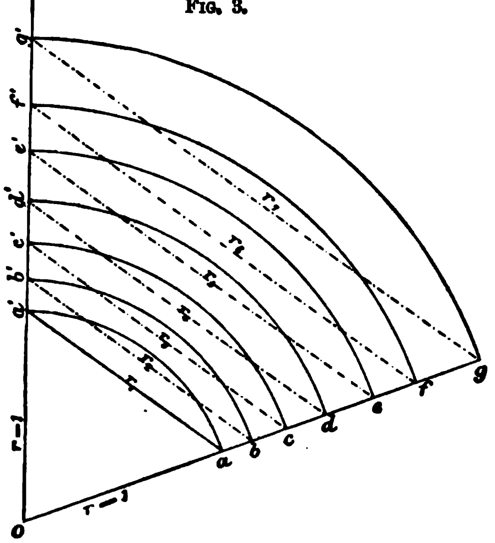
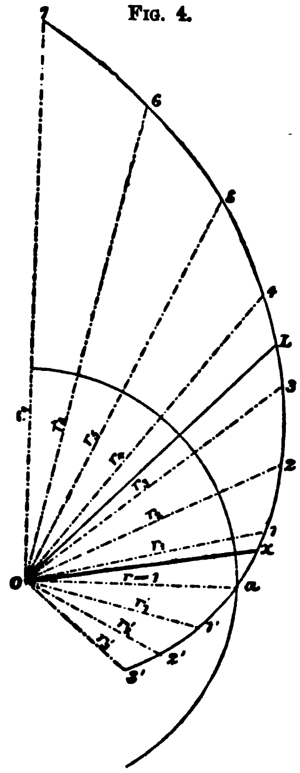
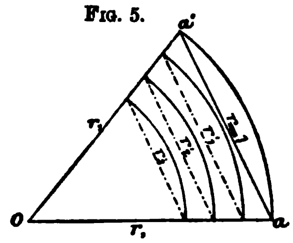


FIG. 4.



spiral. To continue this curve the other way, i. e. *below* the unity radius O a, draw the radii O 1', O 2' .... at equal angular intervals a, and obtain their lengths by constructing an isosceles triangle O a' a (Fig. 5), of which the two sides O a, O a' are both equal to r_1, and the third side $a\,a'$ equal to unity. Having constructed this triangle, describe an arc with centre O and radius $o\,b$ equal to $a\,a'$ (Fig. 4), the chord r_1' of this arc is the required radius O 1'; draw a second arc with radius O c equal to r_1, then the chord r_2' is the required radius O 2' (Fig. 4); by continuing this process the spiral may be extended at pleasure.

FIG. 5.



The spiral so drawn serves to determine any root of any line. Thus, suppose the 5th root ($\sqrt[5]{l}$) of any line l is required. With O as centre and radii equal to unity and l, cut the spiral in a and L respectively, join O L, and divide the angle L O a into five equal parts: make the angle $a\,O\,x$ equal to one-fifth the angle $L\,O\,a$, and on the same side of $O\,a$ as $O\,L$. Then

$$O\,x = \sqrt{\overline{O\,L}} = \sqrt[5]{l}.$$

In the same way the spiral can be used to raise a line to any power. Suppose the value of a line s, raised to the power 7, or s^7, to be required. Cut the spiral, with O as centre and radii equal to unity and s, in a and 1. Join 1 O, and draw a radius O 7 such that the angle $7\,O\,a$ is equal to $7 \times$ angle $1\,O\,a$. Then

$$O\,7 = \overline{O\,1}^7 = s^7.$$

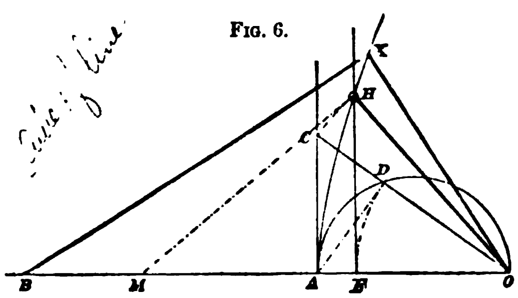
In the figure, to avoid confusion of lines, s has been taken equal to r_1; the construction is, however, quite general.

Instead of the logarithmic spiral the logarithmic curve may be used.

Take any line as axis, from any point on this axis set off abscissæ equal to a series of numbers sufficiently near to each other read off from any convenient scale. At the extremity of each abscissa draw ordinates equal to the logarithms of the respective numbers taken from a table of logarithms and read off from the same scale, and through the extremities of these ordinates draw a curve. This curve can then be used in place of a table of logarithms.

Thus, to obtain the product of a series of lines, set them off from the origin along the axis, and draw the corresponding ordinates; the *sum* of these ordinates is the logarithm of the product, and the abscissa corresponding to an ordinate equal to this sum is the required product.

FIG. 6.



To raise a line l to any power n, set l off along the axis from the origin, and draw the corresponding ordinate, and obtain the abscissa corresponding to an ordinate equal to n times the ordinate of the abscissa l.

Conversely, to extract the n^{th} root of a line l, set off l from the origin along the axis, and obtain the abscissa corresponding to an ordinate of length $\frac{1}{n}^{th}$ of the ordinate of the abscissa l.

The cube root of a given line may be found by means of a special curve drawn as follows:

Make O A (Fig. 6) equal to unity, and on O A describe a semicircle. At A erect a perpendicular to O A. Take any point C on this perpendicular; join C O, cutting the semicircle in D. Make $O\,F = O\,D$, and erect a second perpendicular at F. Make $O\,H = O\,C$. Then H is a point on the curve, and any number of points can be similarly obtained.

Draw H M at right angles to H O;

Then $$O\,H^2 = O\,F \,.\, O\,M.$$

Now the triangles O A C and O A D are similar.

Hence

$$\frac{OA}{OC} = \frac{OD}{OA}; \text{ or, } \frac{1}{OC} = \frac{OD}{1}; \text{ or, } OD.OC = 1.$$

Thus

$$OF.OH = 1, \text{ and } OF = \frac{1}{OH}.$$

$$OH^2 = \frac{1}{OH}.OM, \text{ and } OH^3 = OM.$$

Therefore

$$OM = \sqrt[3]{OH}.$$

The curve can be used as follows:—

From O, with radius equal to O X = l the line of which the cube root is required, cut the curve in X; draw X B perpendicular to O X.

Then as above:—

$$OB = \sqrt[3]{OX}.$$

4. **Division.**—To divide one line by another, or to obtain the value of $\frac{a}{b}$, it is merely necessary to obtain a fourth proportional x, to $\frac{a}{b}$, and unity.

Then

$$\frac{x}{1} = \frac{a}{b}; \text{ or, } x = \frac{a}{b}.$$

5. **Fractions.**—If it is required to add a number of fractions together, or to obtain the value of $\frac{a}{b} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots.$, the fractions must first be reduced to a common denominator, d. Draw two axes, O X, O Y (Fig. 7), at right angles to each other. From O set off O D along O X, equal to the assumed common denominator d; on the same axis set off O b, O b_1, O b_2. . . . , equal to the respective denominators b, b_1, b_2 . . . . of the fractions to be added, and along O Y set off O a, O a_1, O a_2. . . . respectively equal to a, a_1, a_2 . . . . , the numerators. Join $a\,b$, $a_1\,b_1$, $a_2\,b_2$ . . . . , and draw through D the lines D y, D y_1, D y_2 . . . . respectively parallel to $a\,b$, $a_1\,b_1$, $a_2\,b_2$ . . . .

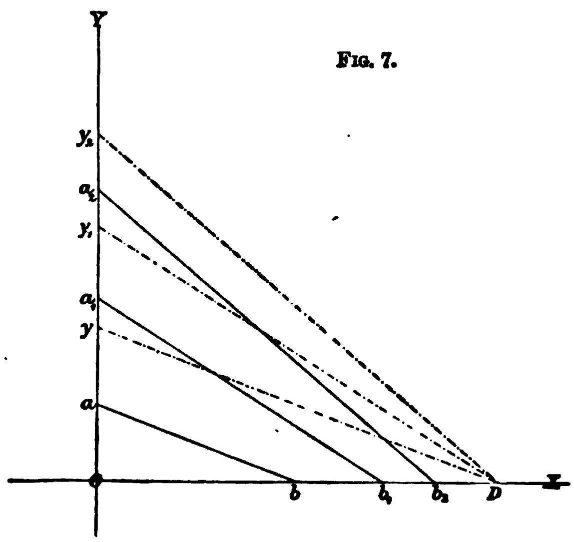


FIG. 7.

Thus

$$\frac{Oa}{Ob} = \frac{a}{b} = \frac{Oy}{OD}, \text{ or } \frac{a}{b} = \frac{Oy}{d},$$

$$\frac{Oa_1}{Ob_1} = \frac{a_1}{b_1} = \frac{Oy_1}{OD}, \text{ or } \frac{a_1}{b} = \frac{Oy_1}{d},$$

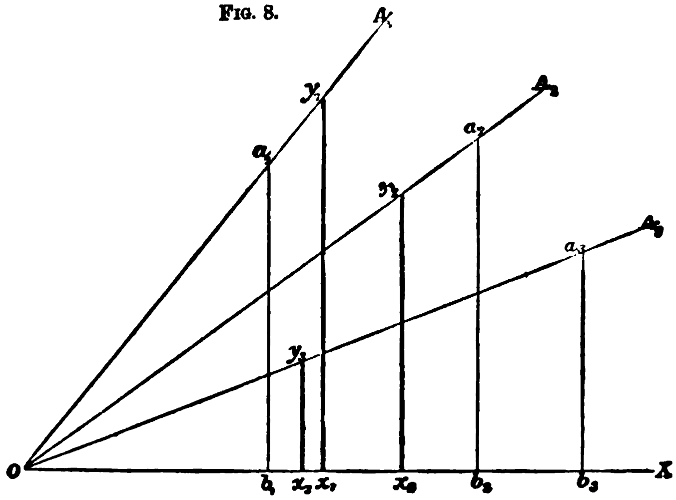
and so on ; hence $$\frac{a}{b}+\frac{a_1}{b_1}+\frac{a_2}{b_2}\ldots\ldots = \frac{O\,y + O\,y_1 + O\,y_2\ldots\ldots}{d}.$$

Therefore, if S is the sum of the lines $O\,y$, $O\,y_1\ldots\ldots$ a fourth proportional to d, S, and unity will be a line representing the required sum of the given fractions.

If any of the fractions of the given series have a *negative* sign, they should be separately reduced to the same common denominator d, and the operation can conveniently be performed by setting the numerators along O Y from O in the *opposite direction* to the numerators of the positive fractions. Then the sum of the numerators of the reduced negative fractions must be subtracted from the sum of the numerators of the reduced positive fractions.

To obtain the product of a series of fractions $\frac{a_1}{b_1}\times\frac{a_2}{b_2}\times\frac{a_3}{b}\times\ldots\ldots$ the method of § 1 can be employed in a slightly modified form. From O set off along any axis $O\,x$

FIG. 8.



(Fig. 8) the distance $O\,b_1$, $O\,b_2$, $O\,b_3\ldots\ldots$ respectively equal to the denominators b_1, b_2, $b_3\ldots\ldots$ of the given fractions; at $b_1\,b_2$, $b_3\ldots\ldots$ draw ordinates $b_1\,a_1$, $b_2\,a_2$, $b_3\,a_3$ respectively equal to a_1, a_2, $a_3\ldots\ldots$ the numerators; through a_1, a_2, $a_3\ldots\ldots$ draw the lines $O\,A_1$, $O\,A_2$, $O\,A_3\ldots\ldots$

Along O X set off $O\,x_1$, equal to unity, and draw an ordinate $x_1\,y_1$ cutting $O\,A_1$ in y_1; set off $O\,x_2$ along O X equal to $x_1\,y_1$, and draw an ordinate $x_2\,y_2$, cutting $O\,A_2$ in y_2.

Then $$\frac{x_1\,y_1}{1} = \frac{a_1}{b_1} \text{ and } \frac{x_2\,y_2}{x_1\,y_1} = \frac{a_2}{b_2}.$$

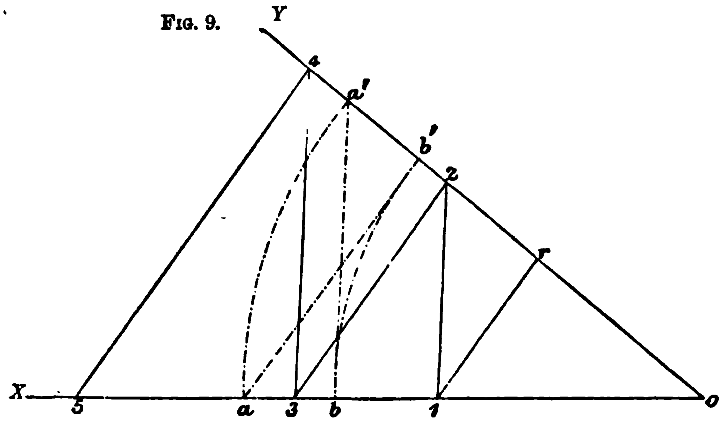
Hence $$x_2\,y_2 = \frac{a_1}{b_1}\times\frac{a_2}{b_2}.$$

and similarly $$x_3\,y_3 = \frac{a_1}{b_1}\times\frac{a_2}{b_2}\times\frac{a_3}{b_3}.$$

Proceeding in this way the product of any number of fractions can be obtained.

To raise a fraction to any power, or to determine the length of a line representing the expression $\left(\frac{a}{b}\right)^n$, draw two lines, O X, O Y (Fig. 9), at any angle, with O as centre describe two arcs, $a a'$, $b b'$ with radii respectively equal to a and b, join $a b'$, $a' b$.

Fig. 9.



From O set off along O Y any arbitrary length O r, and, starting from r, draw a series of lines r 1, 1 2, 2 3, 3 4 .... alternately parallel to $b' a$ and $b a'$.

$$\frac{O1}{Or} = \frac{Oa}{Ob'} = \frac{a}{b} \ldots\ldots (\alpha),$$

$$\frac{O2}{O1} = \frac{a}{b} \ldots\ldots\ldots (\beta).$$

Multiplying the equations α and β together

$$\frac{O2}{Or} = \left(\frac{a}{b}\right)^2 \ldots\ldots (\gamma).$$

Again, $$\frac{O3}{O2} = \frac{a}{b} \ldots\ldots\ldots (\delta).$$

Multiplying γ and δ $$\frac{O3}{Or} = \left(\frac{a}{b}\right)^3,$$

and finally, $$\frac{O6}{Or} = \left(\frac{a}{b}\right)^6.$$

Thus a fraction $\frac{O6}{Or}$ has been obtained equal to $\left(\frac{a}{b}\right)^6$, and having any assigned denominator O r. If O r is made equal to unity, then

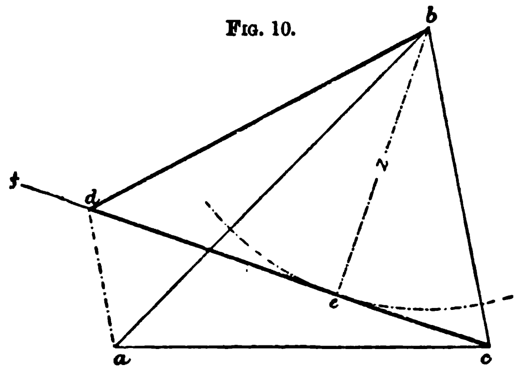
$$O6 = \left(\frac{a}{b}\right)^6.$$

If $b = 1$ and the radius $b b'$ is taken as unity, then O 6 $= a^6$, and the construction may be used in place of that of § 2.

6. **Rectilinear Figures.**—The areas of any rectilinear figures can be expressed by lines. Thus in the triangle $a b c$ (Fig. 10), with any angle b as centre, describe an

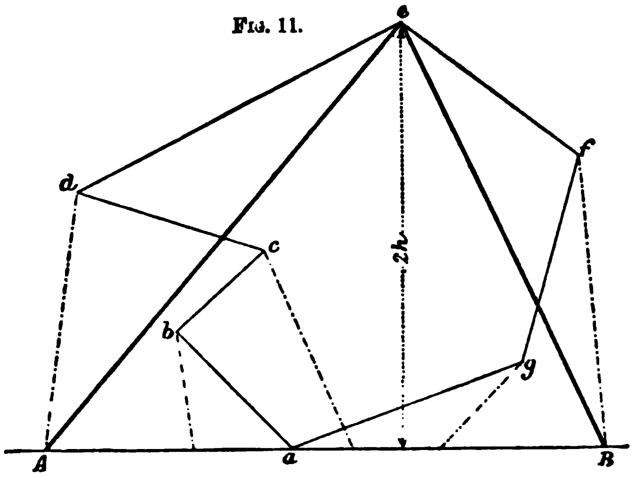
arc with radius be equal to 2 units on the scale to which the triangle is drawn. Through c draw a line cf touching the arc, and through a a line parallel to cb, cutting cf in d. Then the triangle bdc is equal to the triangle bac, and the area of bdc is $\frac{1}{2} be \times dc = dc$, hence dc represents the area of the triangle abc on the same scale as that to which it is drawn. Thus the scale taken was $\frac{1}{4}$ full size, and the length of dc read off on that scale gives 2·95 square inches as the required area. This process is termed the reduction of a triangle to a given altitude: in the above case the altitude is 2 units. If the absolute area of a triangle is not required, but only a line representing that area, any convenient length can be taken for the radius be; thus if a number of triangles are to be represented by lines proportional to their areas, it is necessary to reduce each separately, by the method of Fig. 10, to a triangle having any the same height be, then the bases of the reduced triangles are proportional to their areas. The sum of the bases of the reduced triangles multiplied by half the assumed height gives the sum of the areas of all the triangles.

Fig. 10.



Any rectilinear polygon can be similarly treated, having first reduced it in the usual way to a triangle with vertex at one of its angles and base on a side produced; or it can be reduced directly to a triangle having any given height. Thus the irregular seven-sided polygon $a \ldots . g$ (Fig. 11) is reduced to a triangle A e B, which has a height equal to 2 h.

Fig. 11.



In the case of a trapezium, if m is the arithmetic mean of the parallel sides, d the perpendicular distance between those sides, h the base to which the area of the trapezium is to be reduced; then

$$x . h = m . d, \text{ or } \frac{x}{m} = \frac{d}{h},$$

whence x can be obtained by construction, then x represents the area of the trapezium *reduced to the base h.*

The above construction for the reduction of areas to a given base is required in obtaining the centres of gravity of plane figures.

NOTE.

The foregoing constructions are only directly applicable when the numbers (lines) to be multiplied, divided, &c., are so small that the unit can be taken of workable length, as otherwise the triangles become of such "ill-conditioned" forms that accuracy is not attainable. Well-formed figures are essential in all graphic work. If, for example, in Fig. 1 the lines a_1, a_2 ... represent large numbers, they would be of unmanageable length unless the unit length O 1 were made extremely small. In this case the length O 1 instead of being made of unit length should be taken some convenient multiple of the unit—say 10, 20, 40, 100, or 1000 times the unit.

Suppose O 1 is made equal to m times the unit,

Then

$$\frac{2\,2'}{a_1} = \frac{a_2}{m} \text{ or } 2\,2' = \frac{a_1 \times a_2}{m}$$

$$\frac{3\,3'}{2\,2'} = \frac{a_3}{m} \text{ or } 3\,3' = \frac{a_1 \times a_2 \times a_3}{m^2}.$$

And similarly the last ordinate $= \dfrac{a_1 \times a_2 \times \ldots \times a_n}{m^{n-1}}$.

The result must therefore be read off on a scale the unit of which is $\dfrac{1}{m^{n-1}}$ of the original scale used.

As a numerical example, suppose that the length O 1 is ten times the unit length employed in setting off the lengths a_1, a_2, &c.

Then 2 2′ the product of a_1 and a_2 must be read on a scale $\frac{1}{10}$th smaller than the original one, i. e. each unit of the original scale represents 10 units in the product.

3 3′ the product of a_1, a_2, and a_3 must be read on a scale $\frac{1}{100}$th smaller than the original one, i. e. each unit of the original scale represents 100 units in the second (continued) product, and so on.

CHAPTER II.

COMPOSITION AND RESOLUTION OF FORCES, ETC.

Forces having any Directions in one plane.

7. **Forces acting at a Point.**—In order that a line may represent a force correctly, it is necessary, 1st, that its length should be proportional to the magnitude of the force; 2nd, that its position on the paper should correspond with the line of action of the force; 3rd, that the direction of the force along its line of action, i. e. its "*sense*," should be indicated by an arrow, or by lettering.

Forces which do not act in one plane have to be represented by their projections on two rectangular co-ordinate planes.

If the lines O *a*, O *b* (Fig. 12) represent in magnitude and direction two forces P_1 and P_2 acting at a point O, then the diagonal *r* O of the completed parallelogram O *a r b* represents the resultant of P_1 and P_2 in magnitude and direction. Now the force which, acting in conjunction with P_1 and P_2, would maintain the point O in equilibrium, is equal to R, and has the same line of action, but *opposite sense.* Hence if three forces, P_1, P_2, P_3, acting at a point O (Fig. 13), are in equilibrium, lines drawn parallel to and in the same direction as the forces must form a triangle, and the direction arrows of the forces *point the same way* round this triangle. The order in which the forces are taken in drawing the triangle is immaterial, hence either *b a c* or *b a d* is obtained.

FIG. 12.

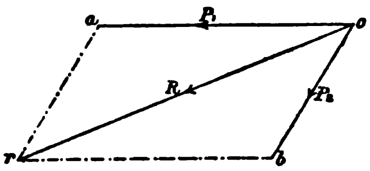


FIG. 13.

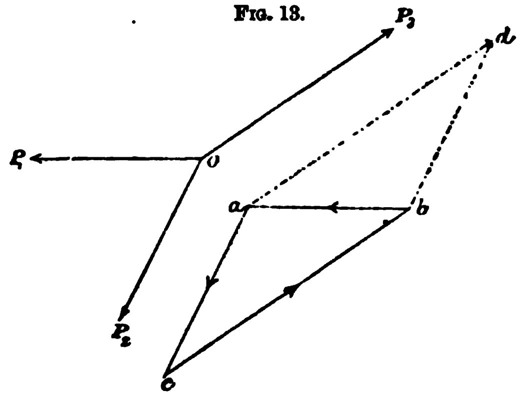
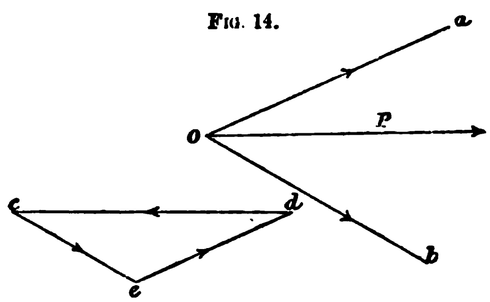


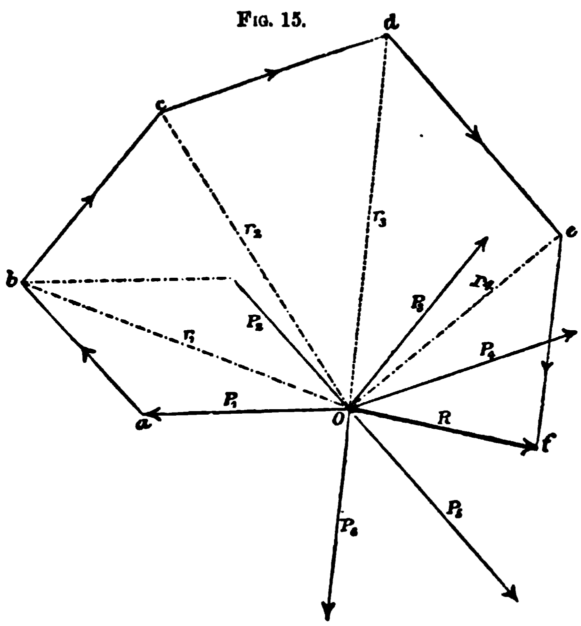
FIG. 14.



The triangle *b a c* or *b a d* can evidently be drawn if the magnitude of one of the forces and the directions of the other two are known. Thus a force P (Fig. 14) can be resolved in two directions O *a*, O *b*; draw *c d* parallel to the direction of P, and representing its magnitude on any scale, and from *c* and *d* draw lines parallel to O *b*, O *a* meeting in *e*, then *c e* and *e d* are the magnitudes of the components of P on the same scale. To obtain the direction arrows of these components affix to *c d* an arrow indicating the reverse direction to that of P, then *c d* represents the force which would maintain equilibrium with the two forces *c e*, *e d*, and the arrows must point the same way all round the triangle *c e d*. This determines the sense of the two components of P.

To obtain the resultant of any number of forces $P_1 \ldots P_6$ acting at a point O (Fig. 15), determine the resultant r_1 of P_1 and P_2; combine r_1 and P_3, thus obtaining the resultant r_2 of P_1, P_2, P_3; continuing the process, the last resultant R is the resultant of all the forces acting at O. Thus to determine the resultant of any number of forces acting at a point, it is necessary to draw a polygon O *a b c d e f* formed of lines drawn successively parallel to, proportional to, and in the same direction as the forces, then the closing line O*f* of this polygon of forces gives the magnitude and line of action of the resultant.

Fig. 15.



Since in the triangle of forces O *e f*, O *f* is the *resultant* of O *e* and *f e*, the direction arrow of O *f* will point in the opposite way round the triangle to the arrows of O *e* and *f e*. This fixes the sense of the resultant R of $P_1 \ldots P_6$.

Since the force which, acting in conjunction with $P_1 \ldots P_6$, would maintain the point O in equilibrium, is equal in magnitude to R, has the same line of action but the opposite sense, it follows that if any number of forces acting at a point are in equilibrium, then lines drawn successively parallel to the lines of action of those forces, in the same direction as the sense of the forces and having lengths proportional to the magnitudes of the forces, must form a closed polygon. Thus if the forces $P_1 \ldots P_5$ (Fig. 16) form a system in equilibrium acting at the point O, the polygon of forces *a* .... *f* must close, the direction arrows pointing the same way all round the polygon.

The *order* in which the forces are taken is immaterial, thus the polygon of forces (Fig. 16) may take the form *a b c d e f* or *a b δ d e ε*.

If *n* forces acting at a point form a system in equilibrium, the *directions of all* and the *magnitudes of* $n - 2$ of them being known, then the polygon of forces can be drawn and the two unknown magnitudes obtained.

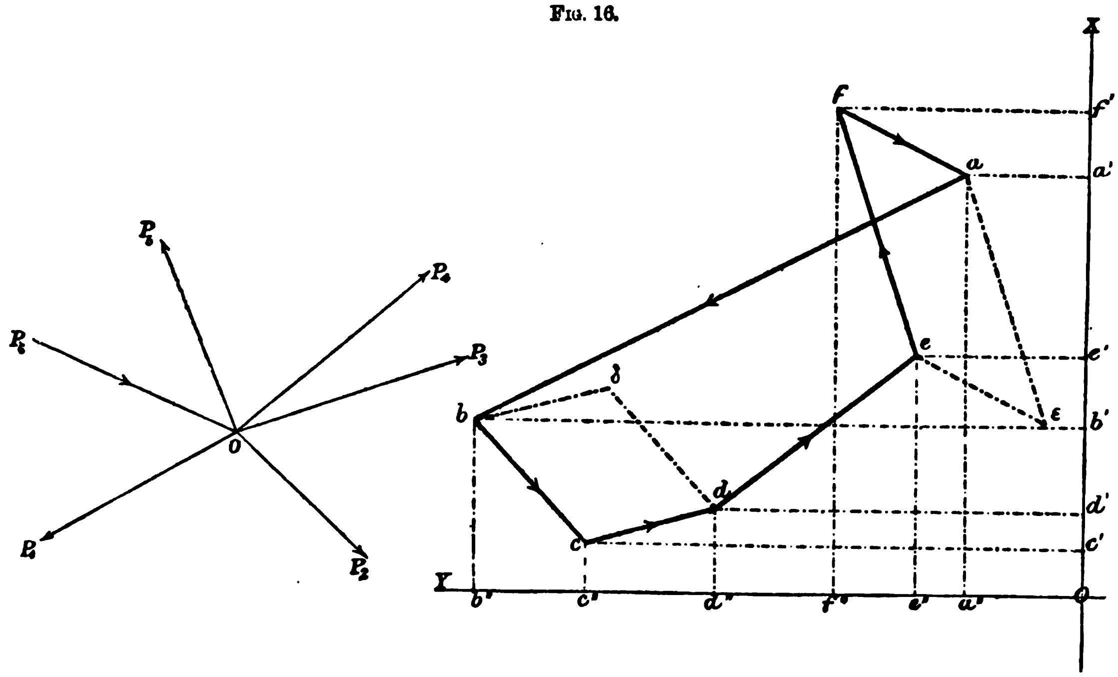
In the case of a system of forces acting at a point, the sole condition of equilibrium is that the polygon of forces should close. This condition corresponds to the analytical condition of equilibrium of forces acting at a point.

If P_1, P_2 .... are any such system of forces, and α_1, α_2 .... the angles made by their directions with any axis, then counting forces acting in one direction as positive and in the other direction as negative

$$P_1 \cos. \alpha_1 + P_2 \cos. \alpha_2 + \ldots\ldots = 0$$
$$P_1 \sin. \alpha_1 + P_2 \sin. \alpha_2 + \ldots\ldots = 0$$

i. e. the sums of the projections on any two perpendicular axes of all the forces acting

FIG. 16.



in one direction are respectively equal to the sums of the projections on the same axes of the forces acting in the other direction. Thus the sum of the projections of *c d*, *d e*, and *e f* (Fig. 16) on the line O X is equal to the sum of the projections of *f a*, *a b*, and *b c*, while the sum of the projections of *b c*, *c d*, *d e*, and *f a* on O Y is equal to the sum of the projections of *e f* and *a b*.

8. **Forces acting in any Direction.**—If three forces only maintain a body in equilibrium, then their directions must pass through a point, hence the sole condition of equilibrium is that stated in the preceding section. If, however, more than three forces act, a further condition must be satisfied.

Suppose the forces $P_1 \ldots . P_5$ (Fig. 17) to act on a body and to maintain that body in equilibrium, and suppose the forces connected by a system of rectilinear bars $A_1 A_2$, $A_2 A_3 \ldots .$ jointed at A_1, $A_2 \ldots .$, and that the system remains in equilibrium, the body having been removed and the jointed bars alone remaining. As a condition of equilibrium of the system, each joint must be separately in equilibrium under the forces acting at it. Thus the exterior force P_1, together with the interior forces or *stresses* S_1 and S_5 in the bars $A_1 A_2$ and $A_1 A_5$ maintain the point

FIG. 17.

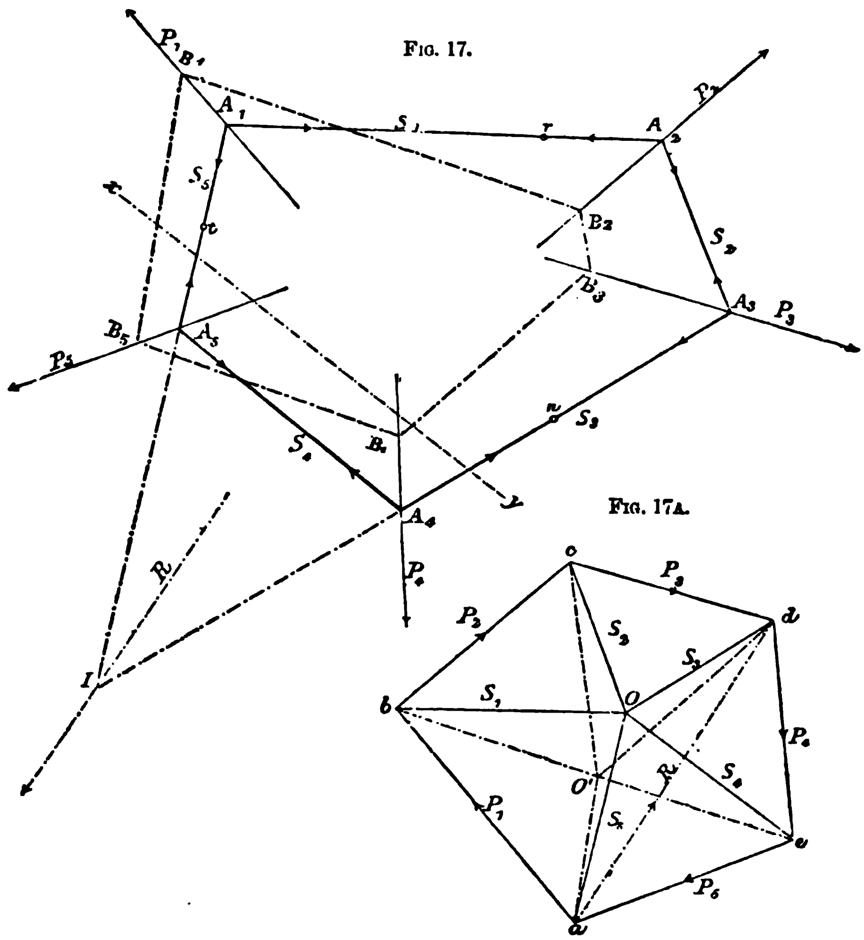


FIG. 17A.

A_1 in equilibrium. Hence P_1, S_1, and S_5 must form a triangle $a b$ O (Fig. 17A); similarly P_2, S_2, and S_1 form a triangle $b c$ O (Fig. 17A) which has one side b O common to the triangle $a b$ O, and so on. Finally P_5, S_5, and S_4 form a triangle $e a$ O which has one side a O common to the first triangle $a b$ O. Thus each joint of the polygonal frame $A_1 \ldots . A_5$ (Fig. 17) furnishes a triangle, each triangle has one side common to *only two* other triangles, and all the triangles make up a *closed* polygon $a b c d e$, the polygon of the exterior forces $P_1 \ldots . P_5$. Further, the vectors drawn from O to the angles of this polygon of forces determine in magnitude, direction, and sense the stresses in the several sides of the polygonal frame $A_1 \ldots . A_5$.

The polygon $A_1 \ldots . A_5$ (Fig. 17), the sides of which are respectively parallel to the vectors drawn from a pole O (Fig. 17A) to the angles of the polygon of forces $a \ldots . e$, is termed the *funicular polygon of the forces $P_1 \ldots . P_5$ with respect to the pole O.*

Reasoning conversely to the above, it will be evident that if a system of forces acting on a body is in equilibrium, then the polygon of forces $a \ldots e$ must close, and if the angles of this polygon are joined to a point O, and starting at any point t a polygon is traced whose angles $A_1 \ldots A_5$ are on the lines of action of the forces, and whose sides are respectively parallel to the vectors drawn from O, this second polygon must close also.

If the funicular polygon $A_1 \ldots A_5$ of the forces $P_1 \ldots P_5$ with respect to any pole O closes, then the funicular polygon $B_1 \ldots B_5$ of the same forces with respect to any other pole O′ closes also.

The conditions of equilibrium of a system of forces acting in any directions on a rigid body are, therefore,

1st. The polygon of forces must close.

2nd. The funicular polygon of the forces with respect to any pole must close.

This second condition corresponds to the analytical condition, that the sum of the statical moments of a system of forces in equilibrium, about any point in their plane, is zero; i. e. if R_1, $R_2 \ldots .$ form a system of forces in equilibrium, then $r_1 r_2 \ldots .$ being the perpendicular distances of their lines of action from any point in their plane,

$$R_1 . r_1 + R_2 \, r_2 + \ldots \ldots = 0$$

reckoning moments which tend to cause rotation in one direction as positive, and in the other direction as negative.

9. **Properties of the Funicular Polygon.**—The funicular polygon (Fig. 17) was drawn starting from the point t; any other point might, however, have been taken as origin. Hence there is an infinite number of funicular polygons having the same pole but different starting points. Similarly there is an infinite number of funicular polygons having the same origin but different poles.

If one of the forces P_1 (Fig. 17) be supposed removed, and a force $-P_1$ having the same magnitude and line of action but the *opposite sense* to P_1, this new force will evidently be the resultant of the remaining forces P_2, P_3, P_4, P_5.

Now suppose that the magnitudes and directions of the four forces $P_2 \ldots P_5$ were known, the magnitude and direction of their resultant $-P_1$ being unknown. Draw the polygon *bcdea* (Fig. 17A) of the four known forces, then the closing side *ab* gives (§ 7, p. 11) the magnitude of the resultant of $P_2 \ldots P_5$. The direction arrow of this resultant will point the *opposite* way round the polygon to the arrows of the four known forces. All that is required, therefore, is a point on the line of action of the resultant, then a line through this point parallel to ba gives the position of the resultant, its magnitude and direction being already known. Taking any point r (Fig. 17) as origin, draw the funicular polygon $r\,A_2\,A_3\,A_4\,A_5\,t$; if then the two last sides A_2r, A_5t of this funicular polygon *coincide*, the forces $P_2 \ldots P_5$ are in equilibrium, and do not admit of a resultant; if A_2r, A_5t are *parallel*, the forces $P_2 \ldots P_5$ do not

Fig. 18.

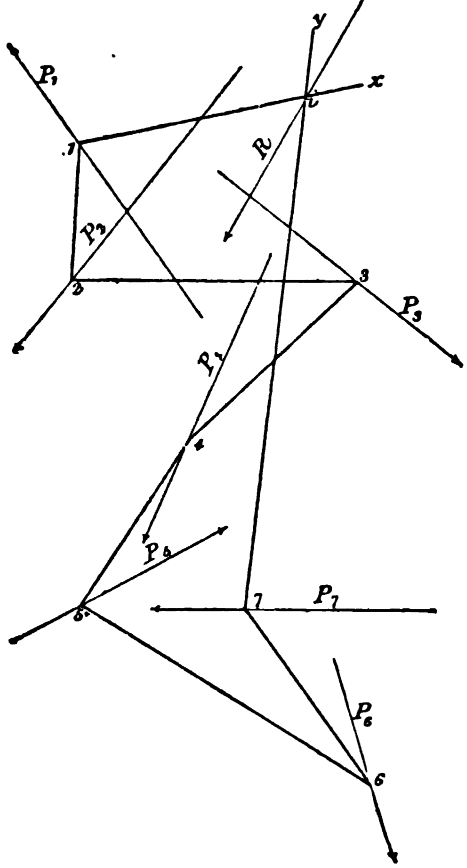
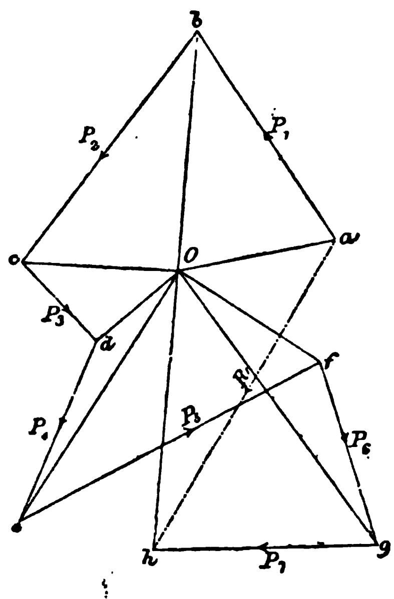


Fig. 18A.



admit of a resultant, but may be reduced to a couple in an infinity of ways; if, however, A_2r, A_5t produced *intersect* in A_1, then A_1 is a point on the line of action of the resultant of $P_2 \ldots P_5$, and that resultant is fully determined.

Thus, to obtain the resultant of a number of forces it is necessary: 1st, to draw the polygon of those forces in order to obtain the magnitude, direction, and *sense* of the resultant; 2nd, to draw the funicular polygon of the forces with respect to any pole in order to obtain a point on the line of action of the resultant.

The above process has been carried out in the case of the forces $P_1 \ldots\ldots P_7$ (Fig. 18), and the magnitude and direction of their resultant R thence obtained.

In Fig. 17 any number of the forces $P_1 \ldots P_6$ can be replaced by their resultant without disturbing equilibrium. Suppose the funicular polygon to be cut by the section plane $x\,y$, then to restore equilibrium a force must be applied equal to the resultant of the stresses S_5 S_3 in the sides A_1 A_5, A_3 A_4. The line of action of this resultant must pass through I, the intersection of A_1 A_5, A_3 A_4 produced, and since the forces R, S_5 and S_3 acting at I must be in equilibrium, these forces must form the triangle $d\,a\,O$ (Fig. 17A), of which the side $a\,d$ gives the magnitude and direction of R. The arrow indicating the sense of R must (§ 7, p. 10) point in the opposite way round the triangle $d\,a\,O$, to the arrows of the sides $a\,O$, $O\,d$, the direction of which can be ascertained by inspection, from the triangles $a\,O\,e$, and $e\,O\,d$ corresponding to the joints A_5 and A_4 respectively: hence R is completely determined.

Thus the funicular and force polygons can be employed to determine the partial resultant of any number of adjacent forces in a system, the polygon of forces furnishing the magnitude, direction, and sense of this partial resultant, while the intersection of those sides of the funicular polygon between which the lines of action of the forces lie, gives a point on its line of action.

Suppose that in Fig. 17 the magnitudes of two of the forces, e. g. P_4 and P_5, are unknown, their lines of action being given. Starting from a (Fig. 17A) the polygon ($a\,b\,c\,d$) of the remaining forces P_1, P_2, P_3 is drawn, then lines from d and a parallel to the given lines of action of the unknown forces determine e; and hence the magnitudes, directions, and sense of these forces are given by the lines $d\,e$ and $e\,a$. Starting from t (Fig. 17) draw the funicular polygon t A_1 A_2 A_3 n with respect to any pole O; produce the extreme sides of this polygon to cut the given lines of action in A_5 and A_4, join A_5 A_4; then A_5 A_4 will be parallel to O e. Now if all the forces are parallel, the polygon of forces becomes a straight line, and since in that case $d\,e$ and $a\,e$ would not intersect, the only way of obtaining e is by drawing a vector from O parallel to A_5 A_4, the closing line of the funicular polygon. This will be noticed hereafter in determining the reactions of supports.

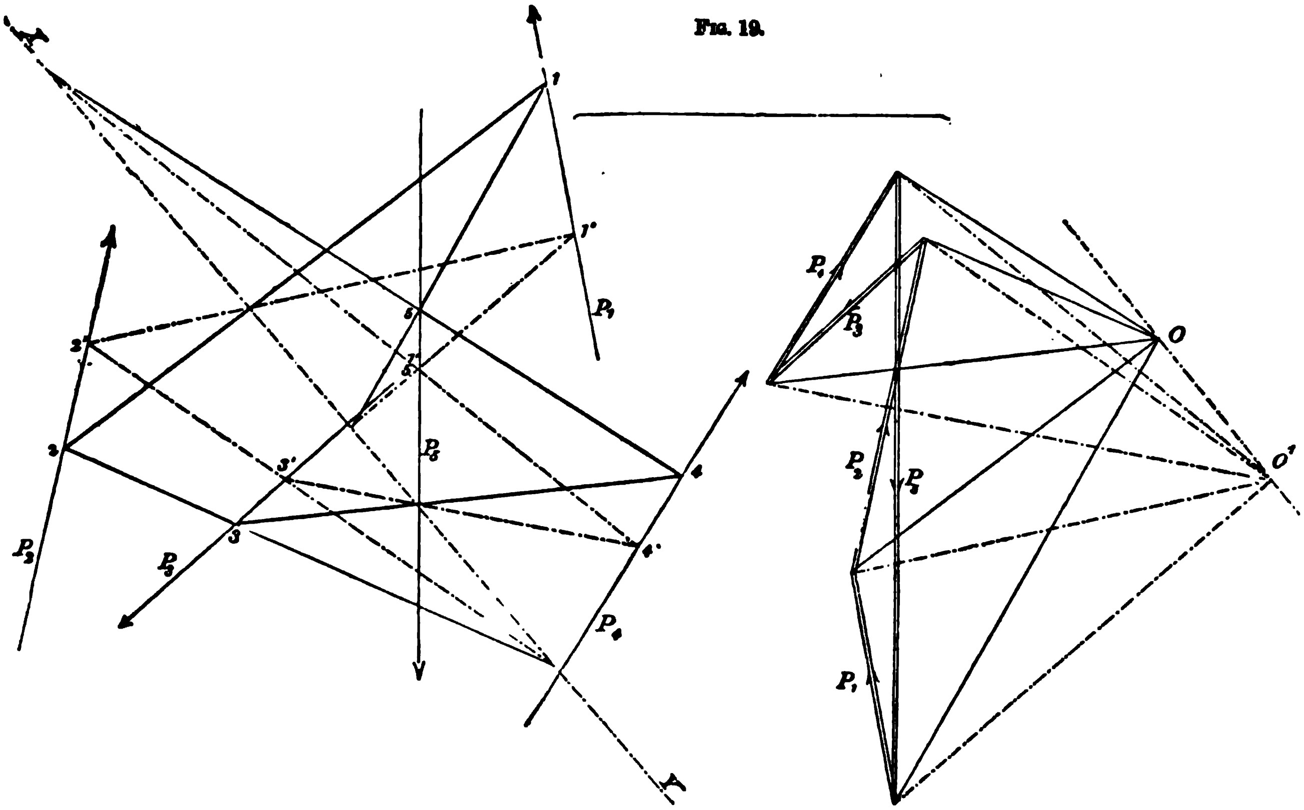
One other remarkable property of funicular polygons remains to be noticed.

If the pole describes a straight line, then the corresponding sides of the successive funicular polygons with respect to successive positions of the pole will all intersect in a straight line, and this straight line is parallel to the locus of the pole.

This is shown in Fig. 19, where the corresponding sides 1 2, 1^1 2^1; 2 3, 2^1 3^1, &c., of two funicular polygons 1... 5, 1^1 ... 5^1 of the forces $P_1 \ldots P_5$ drawn with respect to two different poles O, O^1 intersect on the straight line X Y, and this line is parallel to the line O O^1.

A weightless string suspended at its two extremities to fixed points, and strained by forces distributed at intervals along its length, takes the form of the funicular polygon; the term is, however, used independently of its derivation, and may be equally applied, whether the sides of the polygon to which it refers are in tension or pressure, provided that the polygon conforms to the conditions above described. In the latter case the funicular polygon is sometimes termed a *linear arch*.

FIG. 19.

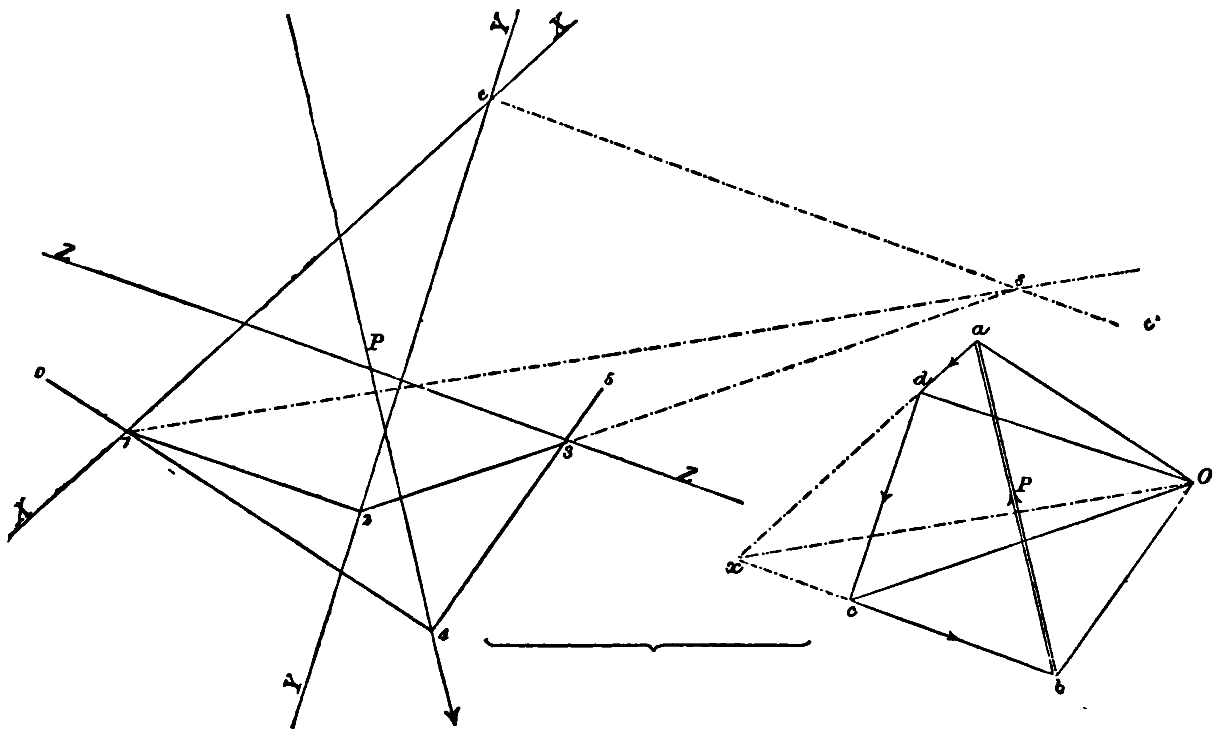


10. Resolution of a Force in Three Directions.—Suppose that the force P (Fig. 20) is to be resolved in three given directions, X X, Y Y, Z Z. If, then, the direction of P is supposed to be reversed, then the reversed force P and the three required components form a system in equilibrium, and the polygon of forces as well as the funicular polygon of this system must close (§ 9). In Fig. 20 the polygon of forces is commenced by drawing *a b* parallel to the direction of the given force P, and joining *a* and *b* to an assumed pole O. From *a* and *b* draw *a x*, *b x* parallel to the given directions X X, Z Z; join O *x*. It is required to obtain in the polygon of forces a side *d c* parallel to the given direction Y Y, and such that the funicular polygon traced by lines parallel to the four vectors O *a*, O *d*, O *c*, O *b* will close.

Suppose the required component Y Y to be resolved into two components Y_1, Y_2, of which Y_1 coincides in direction with X X, and Y_2 is parallel to Z Z. Then evidently the component Y_1, together with the required component X X, make up the length *a x* in the polygon of forces; and similarly Y_2, together with the required component Z Z, make up the length *b x*. The component Y_2 must act along *t t'* drawn from the intersection *t* of X X and Y Y parallel to Z Z.

Starting from *o* draw the funicular polygon for the four forces X X, Y_1, Y_2, and Z Z; *o* 1 parallel to O *a* is the first side, and since X X and Y_1 have the same line of action, the second side becomes *nil*, the third side is 1 *s* parallel to O *x* and terminated at *s* on the line of action *t t'* of the component Y_2. But the five forces P, X X, Y_1, Y_2, Z Z, are in equilibrium, hence their funicular polygon must close. Produce *o* 1 to cut the line of action of P in 4, and draw 4 5 parallel to *b* O, cutting Z Z in 3, then *s* 3 is

FIG. 20.



the closing line of the funicular polygon. In the polygon of forces draw O *c* parallel to *s* 3, *c d* parallel to Y Y, and join O *d*. In the funicular polygon draw 1 2 parallel to O *d*, and produce *s* 3; this amounts to replacing the two components Y_1, Y_2 by their resultant Y Y, and the lines *s* 3 produced and 1 2 will intersect at 2 on the line of action of Y Y. Thus the complete funicular polygon of the forces X X, Y Y, Z Z, and P reversed, is 1 2 3 4. The polygon of forces determines the sense of the components: affix to *a b* an arrow indicating the reverse sense to that of P, then the arrows point the same way all round the polygon of forces.

In Fig. 21 a special case of the same problem is dealt with. Two of the three directions X X, Y Y are parallel to each other, and at right angles to the direction of the given force, while the third, Z Z, is parallel to the direction of the given force. If the construction above explained is carried out in principle, it will be found that in the polygon of forces the two parallel components *a d*, *d c*, are equal in magnitude but of opposite sense, forming therefore a couple, while the third, *c b*, is equal in magnitude and has the same sense as *a b*, the given force. The figuring of the funicular polygon in Fig. 21 has been made to correspond with that in Fig. 20.

The resolution of a force in *more* than three directions is always indeterminate.

The resolution of a force in three directions is impossible if two of the given

Fig. 21

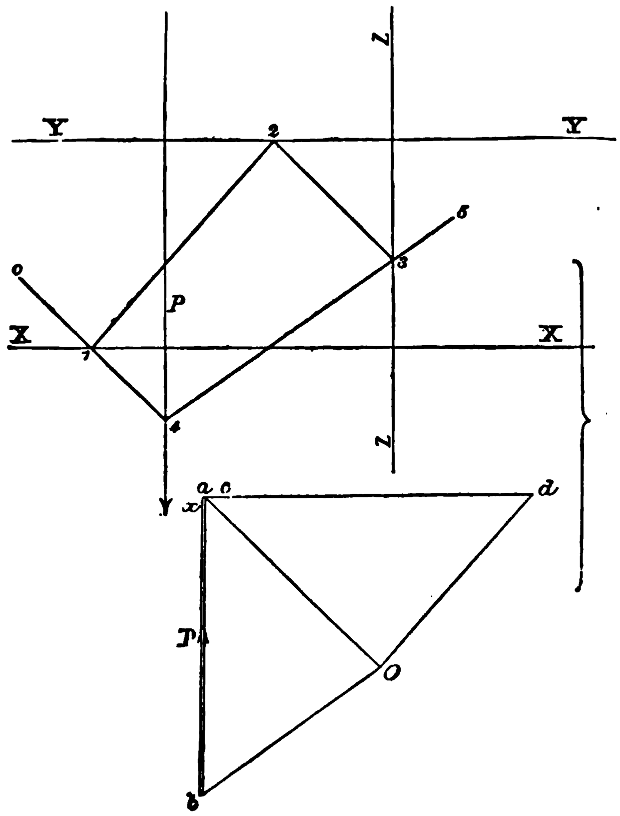
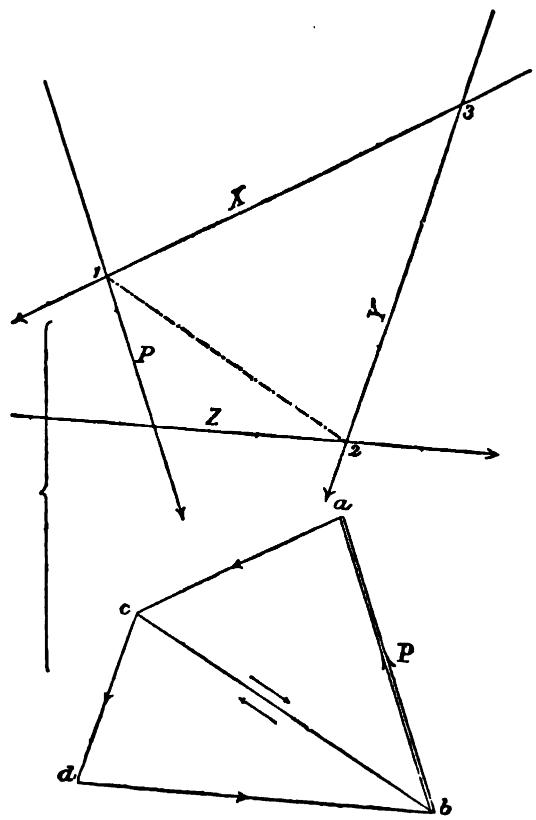


Fig. 22.



directions intersect on the line of action of the given force, and indeterminate if the three given directions intersect on the line of action of the force, or if the three given directions are all parallel to that of the force. It is also obviously impossible if the three given directions are all at right angles to that of the force.

11. **Resolution of a Force in three directions without the direct employment of the Funicular Polygon.**—The construction explained in the above paragraph has been given first as being the most general possible; a simpler method can, however, sometimes be adopted. Thus, in Fig. 22, P is to be resolved in three

directions, X, Y, Z. Produce one of the given directions X to cut the direction of P in 1; join 1 and 2, the intersection of the other two directions, Y and Z. Resolve P first along X and 1 2, and then resolve the component 1 2 along Y and Z. Thus in the polygon of forces draw *a b* parallel and equal to P, and *a c*, *b c* respectively parallel to X and 1 2; affix to *a b* an arrow in the reverse direction to that of P, then the arrows in the triangle of forces *b a c* point the same way round; reverse the arrow of *b c*, and draw *c d*, *b d* parallel to Y and Z respectively, the arrows of *c d*, *d b* point the same way round as that of *c b* reversed.

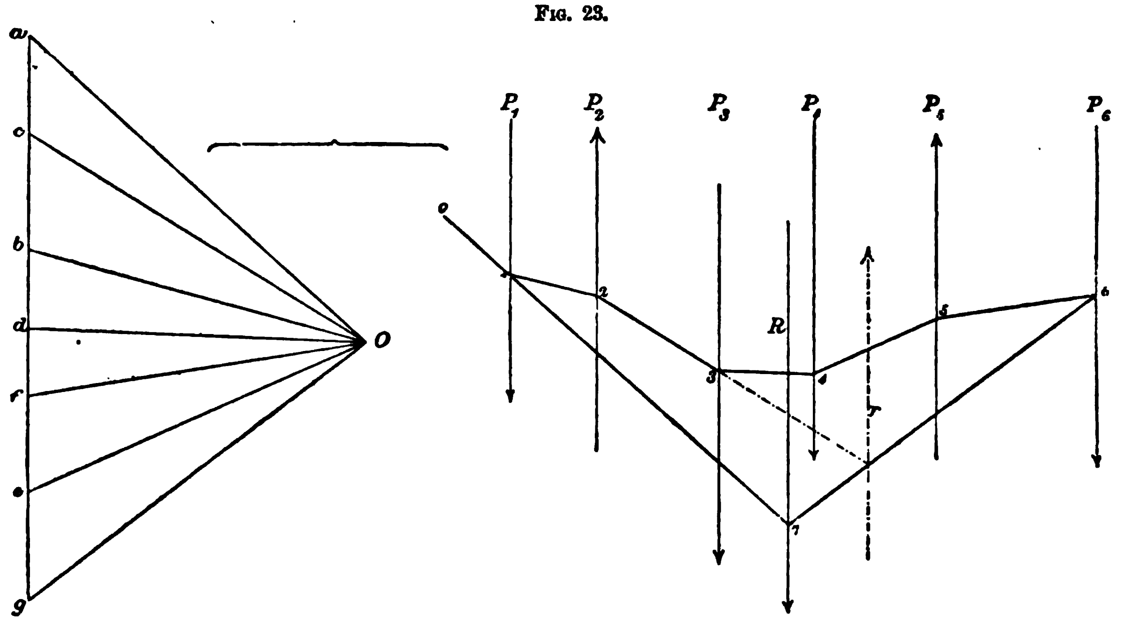
This construction is really a special case of § 10, in which *c*, one of the angles of the polygon of forces, is taken as pole, which allows the lines of action of the forces to be used as the sides of the funicular polygon.

Parallel Forces.

12. **Resultant of a System of Parallel Forces.**— In the case of a system of parallel forces the polygon of forces becomes a straight line, which is sometimes termed the *line of loads;* the principles above laid down are, however, absolutely unchanged.

To obtain R, the resultant of any system of parallel forces $P_1 \ldots . P_6$ (Fig. 23),

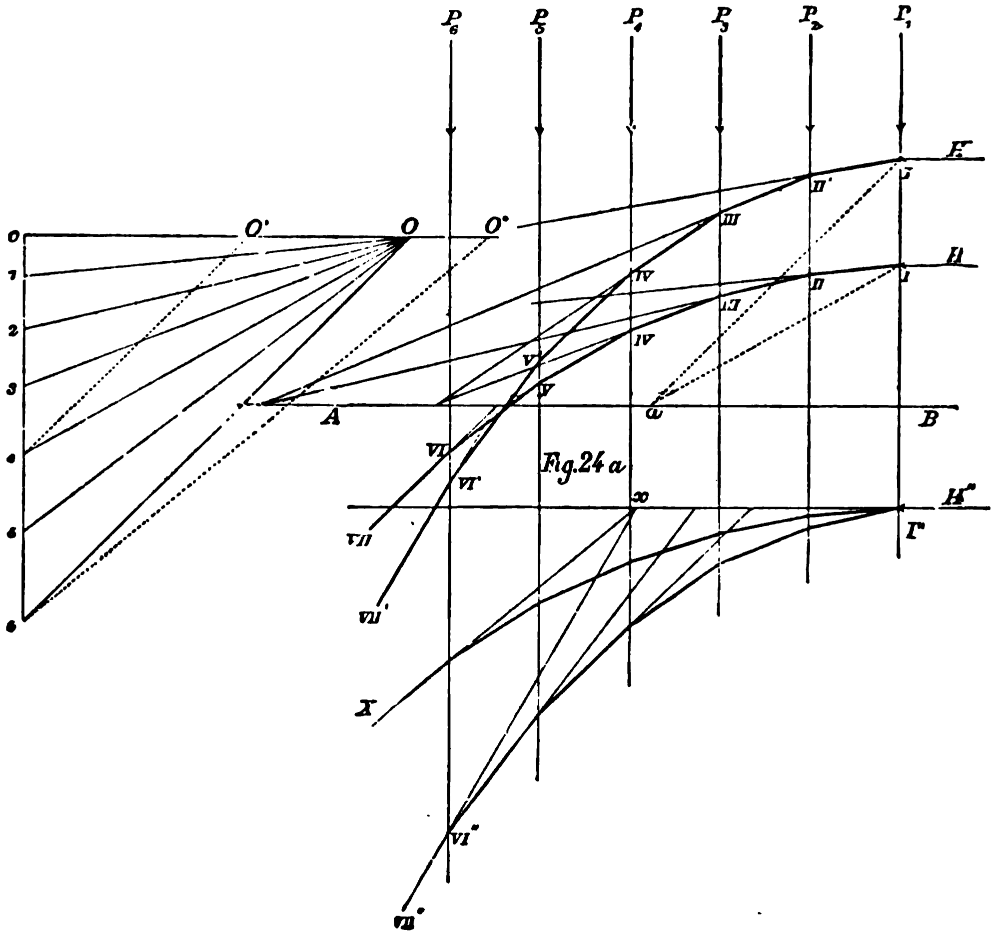
FIG. 23.



draw the polygon of forces *a b c d e f g* starting from *a;* this polygon is a straight line, and *g a*, its closing line, gives the magnitude and sense of the required resultant Take any pole O, and draw the vectors O *a*, O *b* . . . . . O *g;* starting at any point *o*, draw the funicular polygon *o*, 1, ... 6 7, its sides *o* 1, 1 2, &c., being respectively parallel to the vectors O *a*, O *b*, &c. Then, as in § 9, the first and last sides *o* 1, 7 6 of the funicular polygon produced, intersect on the line of action of R: R is therefore fully determined.

Similarly 2 3 and 6 7 produced intersect in *s*, and *s* is (§ 9) a point on the line of action of the partial resultant *r* of $P_3 P_4 P_5 P_6$.

FIG. 24.



Thus, given any system of parallel forces, the resultant of the entire system, or the partial resultant of any number of the forces can be obtained.

If the number of parallel forces is infinite, and their lines of action are infinitely near together, the funicular polygon becomes a curve, termed the *funicular curve.*

The property of the funicular polygon stated in § 9, p. 16, deserves special notice in the case of parallel forces. In Fig. 24, $P_1 \ldots . P_6$ are a system of vertical forces whose lines of action are equidistant. On a load line *o* 6, *o* 1 is made equal to P_1, 1 2 to P_2, and so on. Suppose that it is required to combine these six forces with a horizontal force H: in the polygon of forces draw o O equal to H, and at right angles to the load line; take O as pole, and draw the vectors O 1, O 2, &c. Starting from I, the intersection of the lines of action of P_1 and H, draw the funicular polygon I, II. . . . VII. Let a new horizontal force H′ acting at I′ be substituted for H, the vertical forces remaining as before; make o O′ equal to H′: then (§ 9, p. 16), since the pole has moved from O to O′, the sides of any funicular polygon of the forces with respect to the new pole will severally intersect the corresponding sides of the funicular polygon I. . . . . VII on some line parallel to O′ O. To determine this line, draw I′ II′ parallel to O′ 1, and produce I′ II′ and I II to meet: the line through their meeting point parallel to O′ O is the required line. If, however, the intersection of I′ II′ and I II is at an inconvenient distance, as in Fig. 24, draw lines from I parallel to any vector O 4 of the first force polygon, and from I′ parallel to O′ 4 the corresponding vector of the second force polygon. These lines intersect in a, then A B drawn through a parallel to O′ O is the required line, and corresponding sides of the two funicular polygons intersect on A B as shown.

The above construction has an application in the case of the arch. If $P_1 \dots P_6$ represent the weights of strips of equal breadth of the arch and its surcharge, and H the assumed, or calculated horizontal thrust at the crown, the funicular polygon I. . . . VII is termed the *line of pressure* of the arch. By means of the construction of Fig. 24 a new line of pressure corresponding to any new thrust H′ can at once be derived from the first; or on the other hand, if the line of action of the thrust is known, its magnitude can be so determined that the line of pressure shall pass through any given point. Suppose H″, a thrust of unknown magnitude, to act at I″ (Fig. 24a), and that the line of pressure is to pass through X. Starting at I″, draw the funicular polygon of the forces with respect to any pole O′. Produce the line of action of H″; then evidently, if different values are assigned to H″, or different poles taken along o O, the corresponding sides of the resulting funicular polygons will all intersect on this produced line. Produce VII″, VI″, the last side of the already drawn funicular polygon to meet this line in x. Join X x, and draw 6 O″ in the polygon of forces parallel to X x. Then o O″ gives the magnitude of H″ such that the resulting line of pressure passes through X. This line of pressure is then completed as in Fig. 24a.

13. Resolution of a Force into two parallel components having given lines of action.—In Fig. 25, $b\,a$ is drawn equal and parallel to P, and b and a are joined to any pole O. From any point 1 on the line of action of P draw 1 2, 1 3 respectively parallel to b O, a O, cutting the given directions X and Y in 2 and 3 respectively; join 2 3, then 2 3 is the closing line of the funicular polygon 1 2 3. Draw O c parallel to 2 3, then $b\,c$ is the magnitude of the component X, and $c\,a$ that of Y. If it is remembered that the component X and the stresses in the two sides 2 3 and 1 2 of the funicular polygon must in the polygon of forces form a triangle O *b c*, no doubt can arise as to which of the components *b c* represents.

In Fig. 26 the same construction with identical lettering is shown for the case in which the two given directions are on the *same* side of P. In this case the components cannot both have the same sense.

FIG. 25.

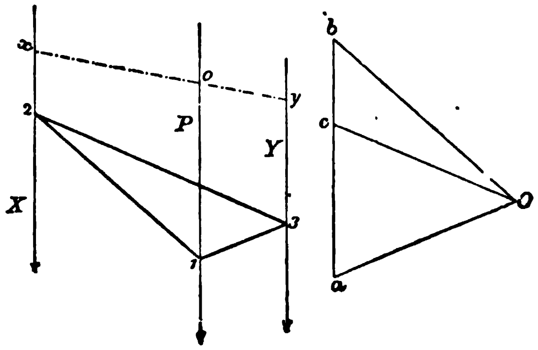
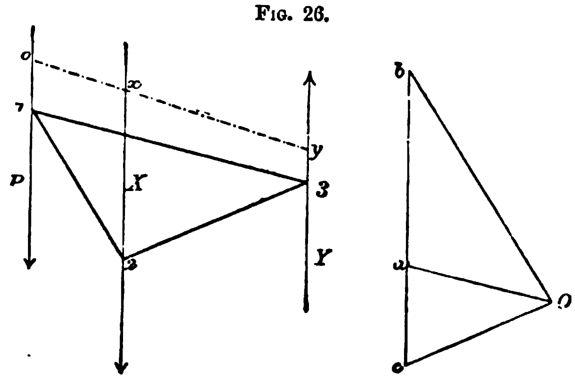
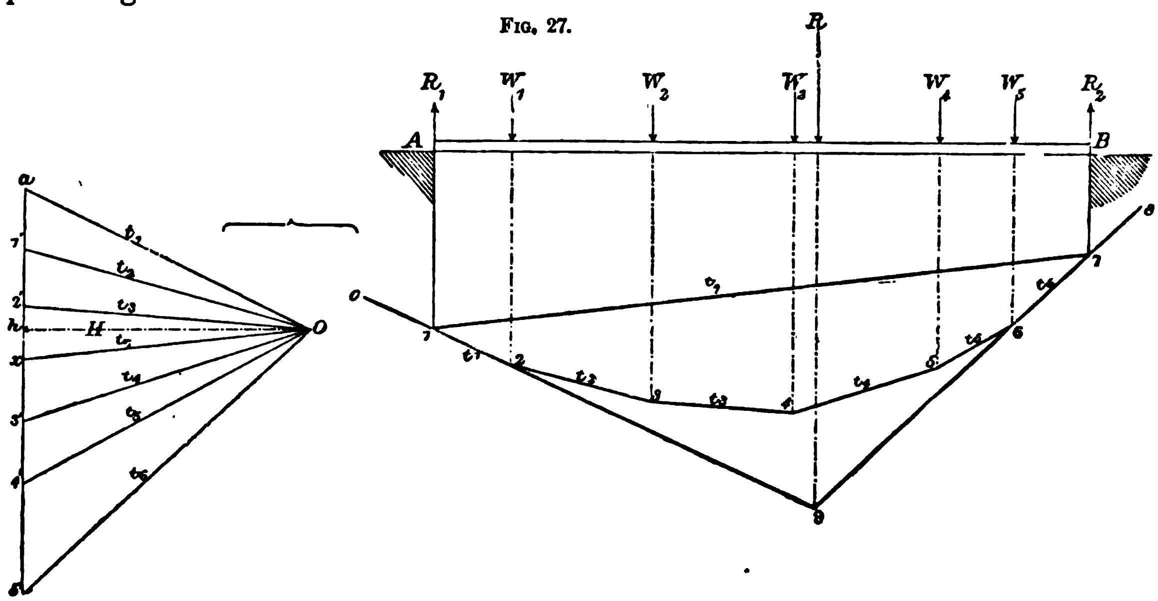


FIG. 26.



In either case the resolution can be effected without the funicular polygon, as follows: Draw a line cutting the three directions at any angle. Divide *b a* in *c* so that $\frac{b\,c}{c\,a} = \frac{o\,y}{o\,x}$, or in the second case produce *b a* to a point *c* such that this proportion holds. If the scale at which *b a* is taken is such that *b a* is not *less* than the perpendicular distance between X and Y, it will only be necessary to place *b a* in between X and Y, producing it in the second case to meet P.

FIG. 27.



14. **Determination of the reactions of the supports at the extremities of a Loaded Beam.**—If a beam A B (Fig. 27) is supported at its two ends, and loaded with the weights $W_1 \ldots W_5$, then, in order that equilibrium may be maintained, the

two supports A and B must furnish reactions $R_1 R_2$, whose resultant is equal to the sum of the weights $W_1 \ldots W_5$. These reactions having been obtained, the supports can be supposed removed, and the forces R_1, R_2, $W_1 \ldots W_5$ can then be dealt with as forming a purely statical system.

It is required, therefore, to determine two forces $R_1 R_2$, having given lines of action, and such that their resultant is equal to the resultant of the given forces $W_1 \ldots W_5$. The polygon of forces and the funicular polygons with respect to any pole, of the whole system $R_1 R_2, W_1 \ldots W_5$ must close. Draw the load line $a\,1'\,2'\,3'\,4'\,5'$, making $a\,1'$ equal to W_1, $1'\,2'$ equal to W_2, and so on, then the whole line $a\,5'$ is equal to $R_1 + R_2$. Assume any pole O, and draw the vectors O a, O 1′, O 2′. . . , then starting at any point o draw the funicular polygon o 2 . . . 6 8 corresponding to the pole O. The sides o 2, 6 8 of this polygon cut the given directions R_1 and R_2 in 1 and 7 respectively, hence 1 7 is the closing side of the funicular polygon. Draw the vector O x parallel to 1 7, then $x a$ is the magnitude of R_1 and $x\,5'$ that of R_2. The resultants of R_1 and R_2 and of $W_1 \ldots W_5$ both pass through 9, the intersection of 1 2 and 7 6 produced, and these resultants are of equal magnitude but opposite sense.

It will be evident that, if the forces are symmetrical and symmetrically distributed about the centre of the beam, the vector O x will bisect $a\,b$, and R_1 is then equal to R_2.

15. **Constant component of stress in the sides of the Funicular Polygon.**— In the case of a system of parallel forces, the component *perpendicular to the direction of the forces* of the stress in the sides of any funicular polygon is constant. In Fig. 27 the stresses $t_1, t_2 \ldots$ in the sides of the funicular polygon are given by the vectors $t_1, t_2 \ldots$ in the polygon of forces. Resolve one of these stresses t_1 into a horizontal and vertical component by drawing O h horizontal and therefore perpendicular to $a\,b$, then O h gives the magnitude of the horizontal and $a\,h$ that of the vertical component. Similarly the horizontal component of any other of the stresses is O h, and hence the horizontal component of all the stresses is the same. If with vertical loads the sides of the funicular polygon are in pressure, as is the case in Fig. 24, the horizontal component is termed the constant *horizontal thrust.* The distance O h is called the *polar distance.*

Moments of Forces.

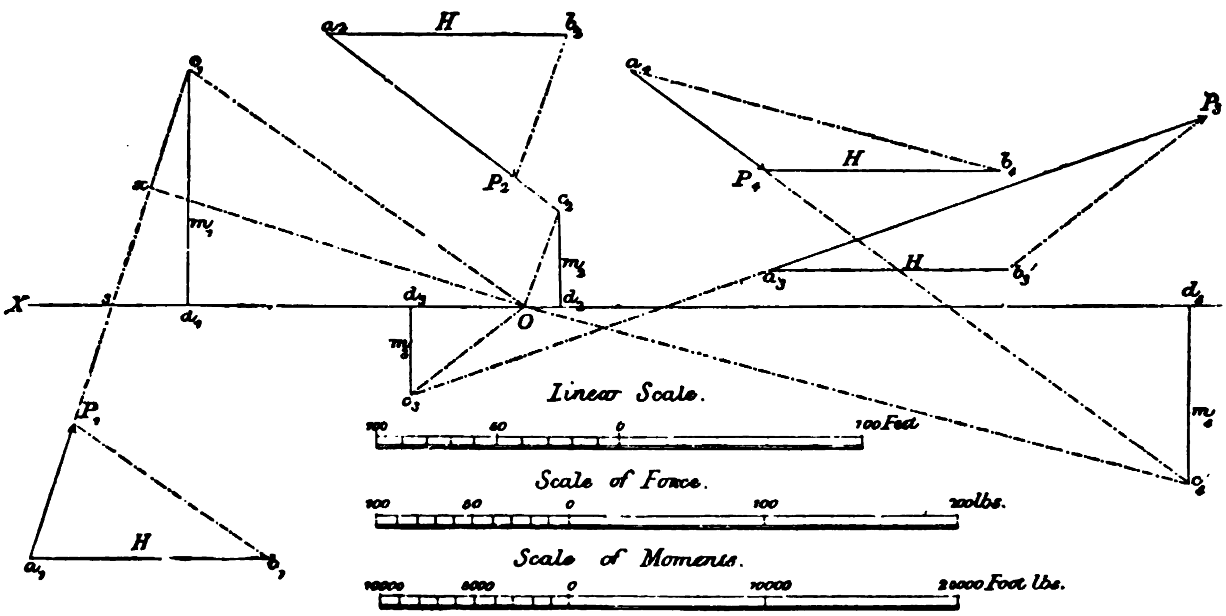
16. **Graphic representation of the Moments of Forces.**—By the moment of a force about a point is meant *the product of the force and the perpendicular let fall on its line of action from the point.*

If a force P act on a rigid body in the plane of the paper, and an *axis perpendicular to this plane* is imagined to pass through the body at any point O, it is evident that the tendency of the force will be to rotate the body round such axis. Hence, although it is customary to speak of the moment of a force about a point, it should always be remembered that moments have relation to some *axis*, and that it is only in the case of forces acting in one plane that the point can be taken instead of that axis. The point is the trace of that axis on the plane of the forces, or in the case of a single force the point about which the moment is taken is the trace of the axis on the plane through the force and perpendicular to the axis.

FIG. 27A.

The moment of a force about any point in its plane is equal to double the area of a triangle whose base represents the magnitude of the force, and whose height is equal to the perpendicular distance of the point from the direction of the force. The moments of a system of forces can, therefore, be represented by a series of rectangles, and if these rectangles are reduced to equivalent rectangles, all of which have equal bases, then evidently a scale of moments can be constructed from which the heights of the rectangles read off give the values of the several moments in terms of any unit of moment adopted. The moment unit is the moment of a unit of force acting at a unit of perpendicular distance, and may be a *foot pound*, a *foot ton*, an *inch ton*, &c., according to the linear and force units adopted.

FIG. 28.



17. Reduction of Moments to a common base.—In Fig. 28 the lines $a_1 P_1$, $a_2 P_2$, ... represent in magnitude, direction, and position forces whose moments about any point O in their plane are to be reduced to a common base.

The linear scale is 80 feet to 1 inch.

The scale of force is 100 pounds to 1 inch.

Thus—

$$a_1 P_1 = 75 \text{ lbs.}$$
$$a_2 P_2 = 120 \text{ ,,}$$
$$a_3 P_3 = 242 \text{ ,,}$$
$$a_4 P_4 = 88 \text{ ,,}$$

Suppose that the common base to which the moments are to be reduced is $H = 100$ feet.

Draw any line X Y through O, and from a_1, a_2 .. draw the lines $a_1 b_1$, $a_2 b_2$, ... parallel to X Y, and equal to the given base H. Join $b_1 P_1$, $b_2 P_2$ .... Through O draw $O c_1$, $O c_2$ ... parallel to $b_1 P_1$, $b_2 P_2$ .... and meeting $a_1 P_1$, $a_2 P_2$ .... produced, if necessary, in c_1, c_2, .. From c_1, c_2 .. drop perpendiculars $c_1 d_1$, $c_2 d_2$, ... on the line X Y. Then these perpendiculars $c_1 d_1 = m_1$, $c_2 d_2 = m_2$, &c., are the required heights of the reduced rectangles, having H as base.

From O drop a perpendicular $O x$, on the direction of $a_1 P_1$, and let s be the point in which $a_1 P_1$ produced cuts X Y. Then the triangles $a_1 P_1 b_1$ and $s c_1 O$ are similar.

Hence
$$\frac{a_1 P_1}{H} = \frac{s c_1}{s O}.$$

But since the triangles $s d_1 c_1$ and $s x O$ are also similar,

$$\frac{s c_1}{s O} = \frac{c_1 d_1}{O x}.$$

Thus
$$\frac{a_1 P_1}{H} = \frac{c_1 d_1}{O x}; \text{ or } a_1 P_1 \,.\, O x = m_1 \,.\, H.$$

Similarly the moment of $a_2 P_2$ about O is $m_2 \,.\, H$, and so on.

The scale of moments must be so drawn that y units on the scale of force equals $y \times 100$ units on the scale of moments; thus on the latter scale 1 inch represents 10,000 foot-pounds.

Reading off the distances m_1 m_2 m_3 m_4 on a scale of moments so constructed, we obtain—

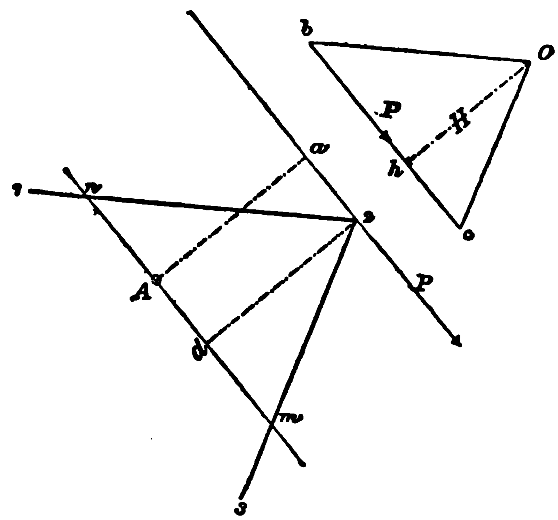
Moment of $a_1 P_1$ = 12,000 foot-pounds.
,, $a_2 P_2$ = 5,000 ,,
,, $a_3 P_3$ = 4,500 ,,
,, $a_4 P_4$ = 9,400 ,,

Moments tending to effect rotation in the direction of the hands of a watch are usually reckoned as positive; hence m_1, m_2, and m_4 are *positive*, and m_3 is a *negative* moment. The sum of the moments, or the total moment, is 21,900 foot-pounds.

18. **Application of the Funicular Polygon to the Determination of the Moment of a Single Force.**—In Fig. 29 P is the force, the moment of which about the point A is required.

Draw $b\,c$ parallel to the direction of P, and equal to it in magnitude. Take a pole O at a distance from $b\,c$ equal to H (any convenient multiple of the linear unit), and draw the funicular polygon 1 2 3 of the force P with respect to the pole O. Through A draw a line parallel to the direction of P, and cutting the sides 1 2, 2 3 of the funicular polygon in n and m respectively. Draw 2 d perpendicular to $m\,n$.

FIG. 29.



Then, since the triangles $m\,2\,n$, $c\,O\,b$ are similar—

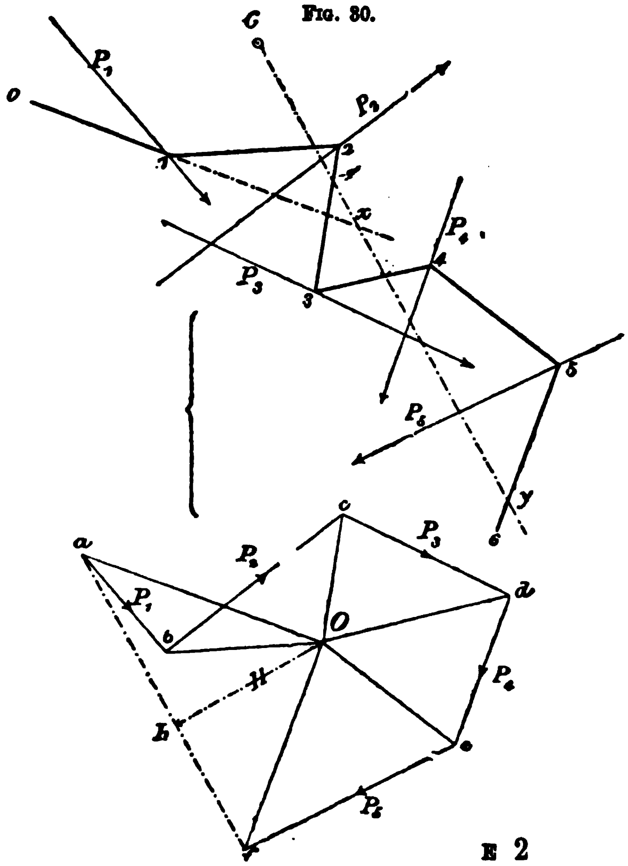
$$\frac{b\,c}{O\,h} = \frac{m\,n}{2\,d};$$

or, $$P \cdot A\,a = m\,n \cdot H.$$

Hence, if a line is drawn through any point parallel to the direction of a force, then the portion of this line intercepted between the two sides of the funicular polygon which meet on the line of action of the force, multiplied by the polar distance, is the moment of the force about the point.

If the polar distance H is made unity on the linear scale, then the length $m\,n$ read off on the scale of force gives the number of moment units. If H is not unity, then the length $m\,n$ read off on the scale of forces must be multiplied by the number of linear units which H represents; the product will be the number of moment units. It will therefore be convenient, if the moments of a system of forces have to be dealt with, to take H a round number of linear units. The multiplication can be performed arithmetically, or by drawing a scale of moments as in the preceding paragraph.

FIG. 30.



19. **Moment of the Resultant of any System of Forces.**—In accordance with the principles of the pre ceding section, the moment about any point C of the resultant of a system of forces $P_1 \ldots P_5$ (Fig. 30) can be

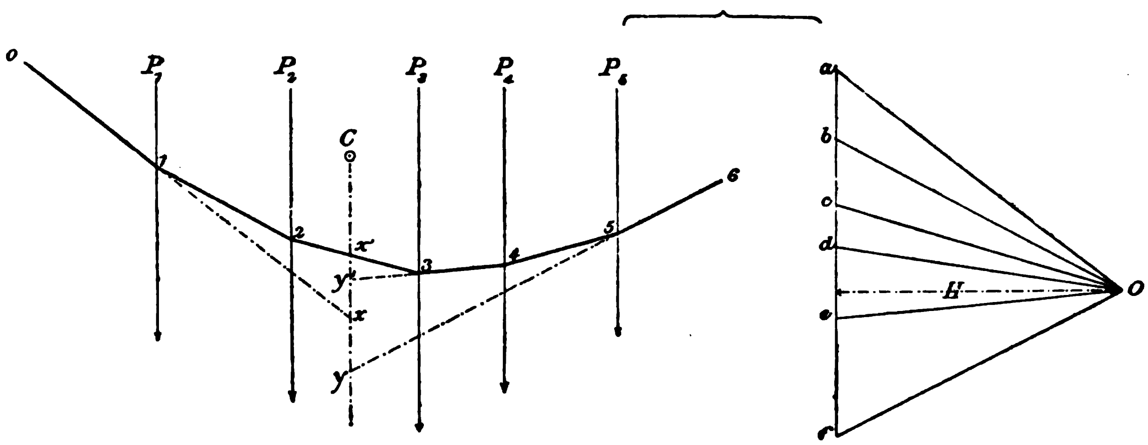
obtained. Draw the polygon $a \ldots f$ of the five forces, then the resultant of the forces is parallel to $f\,a$, the closing line of this polygon. Take a point O, whose distance O h from $a\,f$ is equal to the base H of the required moment, and draw the funicular polygon 0 1 2 3 4 5 6 of the forces $P_1 \ldots P_5$ with respect to O. Then the first and last sides 0 1, 6 5 of the funicular polygon produced will intersect on the line of action of the resultant. Draw through C a line parallel to af, cutting the produced sides 0 1, 6 5 in x and y respectively.

Then the moment of the resultant of the whole system is equal to $x\,y\,.\,\mathrm{H}$, and this moment is equal to the sum of the moments of the separate forces about C.

If the funicular polygon closes, the sides 0 1, 6 5 coincide, and the intercepted distance $x\ y$ disappears. The moment of the resultant is therefore zero, and the forces are in equilibrium. Hence, as has been noticed in § 8, the closing of the funicular polygon corresponds to the analytical condition of equilibrium, that the algebraic sum of the moments about any point is equal to zero.

20. **Moment of the Resultant of Parallel Forces.**—The moment about the point C of the resultant of the five parallel forces $P_1 \ldots P_5$ (Fig. 31) can be obtained

FIG. 31.



in a precisely similar manner. Draw the line of loads af, and also a funicular polygon 0 1 ... 6 with respect to a pole O at a distance H from af. Then the moment of the resultant of the system about C is equal to $x\ y\ .\ \mathrm{H}$.

The moment of the partial resultant of any number of the forces can be similarly obtained. Thus the moment of the resultant of P_4 and P_5 is equal to $y'\,y\ .\ \mathrm{H}$, and the moment of P_3 is equal to $x'\,y'\,.\ \mathrm{H}$.

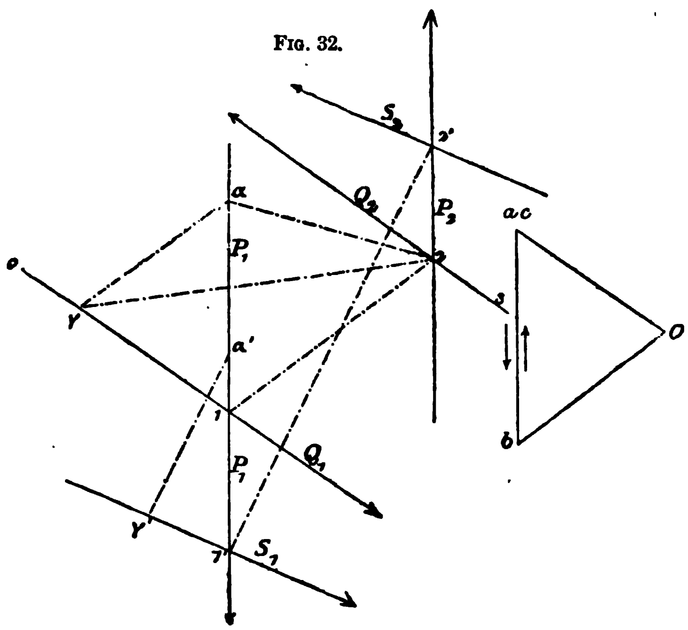
The above construction gives the moment of any force or combination of forces in the form of a product which has one common factor H. This factor is termed the "moment base"; and since the length of H in the polygon of forces may be taken at pleasure, the moments of any number of forces, or the moments of the resultants of any groups of those forces, can, by means of the above construction, be reduced to any common base. The moments of any number of forces having been thus reduced to a common base, their summation reduces itself to the summation of the lengths of certain intercepts by the sides of the funicular polygon on a line through the centre of moments, and parallel to the direction of the forces.

The above has a direct practical application in the determination of the bending moments at the various sections of a loaded beam, and also in obtaining the centre of gravity and moment of inertia of plane figures.

Couples.

21. **Moment of a Couple, &c.**—P_1 and P_2 (Fig. 32) are two equal parallel forces of opposite sense forming a couple. Starting from any point *a*, draw the polygon of forces *a b*, *b c* of the couple, the two sides of which coincide, since the forces are equal and parallel. Taking any point O as pole, draw the funicular polygon 0 1 2 3, of which the two sides 0 1, 2 3 are parallel to each other and to the vector O *a*, while the side 1 2 is parallel to O *b*. Suppose P_1 to be replaced by its two components *a* O, O *b* acting along the lines 0 1, 1 2 respectively, and P_2 to be replaced by *b* O and O *c* acting along the lines 1 2 and 2 3, then the two components acting along the line 1 2 evidently balance each other, and the remaining two components *a* O, O *c* acting along 0 1 and 2 3 form a new couple Q_1 Q_2, which would replace the original couple P_1 P_2.

Fig. 32.



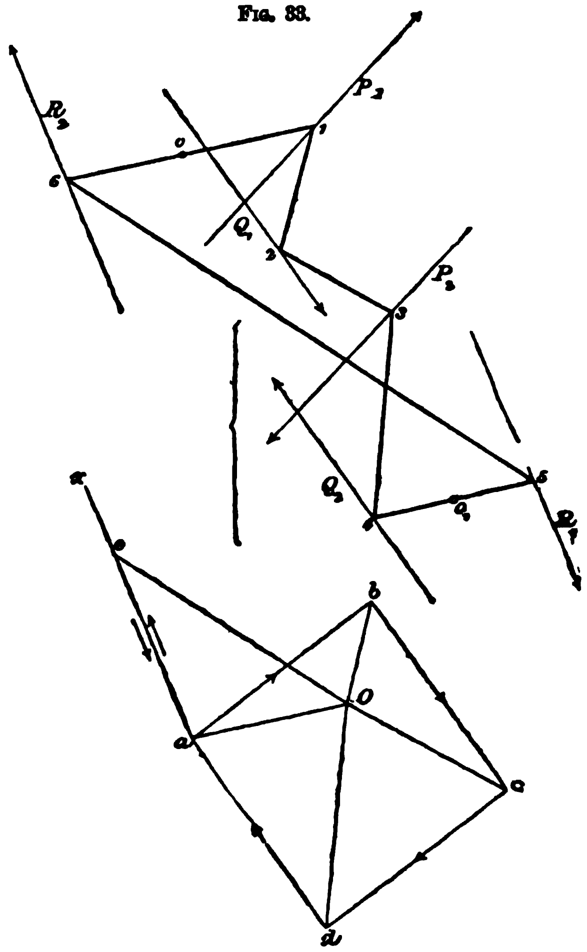
Now, the moment of a couple about any axis perpendicular to its plane is equal to the product of one of its equal forces into the perpendicular distance between their directions. Make 1 *a* equal to *a b* or P_1, and join *a* 2, then the moment of the couple P_1 P_2 is equal to double the area of the triangle 1 *a* 2. From 1 set off 1 γ on 1 0 equal to *a* O or Q_1, then the moment of the new couple Q_1 Q_2 is equal to double the area of the triangle $1 \gamma 2$. Join γa, then the angle $1 a \gamma$ is equal to the angle $a b O$ and γa is parallel to $1\,2$. Hence the triangle $1 \gamma 2$ is equal to the triangle $1 a 2$, or the moment of the couple $Q_1 Q_2$ is equal to that of $P_1 P_2$.

It is evident, therefore, that a given couple can be replaced by an equivalent couple having any given lines of action, provided only that the area of a triangle having a base on the line of action of one of the forces and equal to that force, while its vertex is on the line of action of the other force, remains constant.

Thus, if the original couple $P_1 P_2$ is to be replaced by an equivalent couple having any lines of action $S_1 S_2$ cutting P_1 and P_2 in $1'$ and $2'$ respectively; make $1' a'$ equal to P_1, join $1' 2'$, and draw $a' \gamma'$ parallel to $1' 2'$, then $\gamma' 1'$ is the required magnitude of the equal forces $S_1 S_2$. The sense of these forces must of course be such that the couple tends to effect rotation in the *same direction* as the original couple.

22. **Resultant Couple.**—A number of couples acting in a plane can be replaced by a resultant couple having given lines of action. $P_1 P_2$ and $Q_1 Q_2$ (Fig. 33) are two couples; it is required to replace them by a resultant couple having given lines of action R_1, R_2.

FIG. 33.

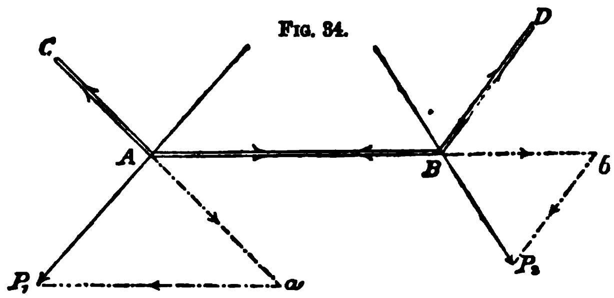


Draw the polygon of forces $a b c d$, making $a b$ parallel and equal to P_1, $b c$ parallel and equal to Q_1, and so on. The figure obtained will be, of course, a parallelogram. Take any point O as pole, and draw the vectors $O a$, $O b$, $O c$, $O d$. Starting at any point 0, draw the funicular polygon $0\,1\,2\,3\,4\,0_1$ of the four forces P_1, Q_1, P_2, Q_2 relative to the pole O. From a in the polygon of forces draw $a x$ parallel to the given directions R_1 and R_2, and in the funicular polygon produce the two last sides $1\,0$, $4\,0_1$ to cut these given directions in 5 and 6 respectively. Join 5 and 6, and from the pole O draw $O e$ parallel to the last side $5\,6$ of the funicular polygon, then $a e$ gives the magnitude of the equal forces R_1, R_2, forming the required resultant couple. The polygon of forces $a b c d a e a$ will give the sense of the required couple.

Attach direction arrows to the side *a e* as shown, the arrows then point the same way all round the polygon, and as both the polygon of forces and the funicular polygon close, the system is in equilibrium; hence the direction arrows of the *resultant couple* must be the reverse of those appended to the side *a e* of the polygon of forces, that of R_1 pointing down, and that of R_2 pointing up.

23. **Interior Forces or Stresses.**—In Fig. 34, C A, A B, B D are three sides of a jointed polygonal frame strained by two forces P_1, P_2 acting at the joints A and B respectively; resolve P_1 along the directions of the bars C A, A B, and P_2 along the directions of A B, B D. The resolution is effected by means of the triangles of forces A P_1 *a*, B P_2 *b*, and the directions of the components are those shown by the arrows (§ 7); then the stresses in the bars will be equal in magnitude but opposite in direction to the resolved components of the exterior forces P_1 . P_2. The directions of these stresses are also shown by arrows, and the bar A B is evidently in tension. Hence, for tension in any bar of a structure the arrow-combination ⟶ ⟵ obtains, and for pressure ⟵ ⟶.

FIG. 34.



CHAPTER III.

RECIPROCAL FIGURES.

24. **Definition of Reciprocity.**—Two plane figures are said to be reciprocal to each other when both fulfil the following conditions:—

a. To every side of one figure there is *one* corresponding side in the other figure.

b. Corresponding sides are parallel, perpendicular, or inclined at some constant angle to each other.

c. To every system of lines meeting in a point of one figure there corresponds a closed polygon in the other figure.

Thus the first of two reciprocal figures can be deduced from the second, or the second from the first, in precisely the same way. The number of sides in two reciprocal figures is the same; and since the question of *size* does not enter into the above conditions, if one figure is reciprocal to another, all figures *similar* to the first will be reciprocal to the second, and *vice versâ*.

25. **Classification of Figures.**—In order that it may be possible for a given figure to fulfil the above conditions, it is necessary that—

1st. The figure can be decomposed into a number of closed polygons such that each side of the original figure is a side of *two only* of these polygons.

2nd. *At least three* lines of the original figure meet at each of its angular points, or vertices.

3rd. Every side of the original figure passes through *at least two* vertices.

In any figure fulfilling these three conditions, if s is the number of sides, v the number of vertices, and p the number of closed polygons into which it is possible to decompose it—

Then it can be shown that $p + v - s = 2$.*

Plane figures can be divided into the following classes:—

Deformable Figures, whose angles admit of variation without alteration in the length of the sides. Such figures are not therefore sufficiently determined if their sides only are known.

Indeformable Figures, whose angles are determined if the lengths of their sides are known.

The latter class can be subdivided into *Strictly Indeformable Figures* and *Figures having surplus lines.*

A figure is strictly indeformable if it contains only the exact number of sides sufficient to determine it. Such a figure therefore becomes deformable if any one side is suppressed. A figure is said to have one, two, or three *surplus lines* if it contains one, two, or three more sides than are absolutely necessary to determine it.

If s is the number of sides, v the number of vertices of a strictly indeformable figure, then, since every vertex is determined by two sides, and two only, there will be, *exclusive of any side a b,* $2(v - 2)$ sides; the total number of sides is therefore $2(v - 2) + 1$. Draw n new lines on the figure joining vertices already determined. These new lines will be surplus lines, that is to say, they are not required to determine the figure, and it must be possible to express them in terms of the other sides of the figure. Hence, in the case of a figure having n surplus lines, the total number of sides is $2(v - 2) + 1 + n$. Again, in the original strictly indeformable figure, suppose n' of the lines removed, then the total number of sides remaining will thus be $2(v - 2) + 1 - n'$; and it is impossible to construct the figure if only these remaining sides are known, for there will be an infinite number of figures having these sides but different angles. Hence, in order that the new figure should become strictly indeform-

* Levy, 'La Statique Graphique,' Chap. III. § 27

able, it is necessary that it should fulfil n' geometrical conditions. The results of the above reasoning may be expressed as follows.—

In any deformable figure, $s < 2v - 3$.
" *strictly indeformable figure,* $s = 2v - 3$,
" *figure having surplus lines,* $s > 2v - 3$.

26. **Conditions of possible Reciprocity.**—A figure containing *one* surplus line admits of only one reciprocal.

A strictly indeformable figure admits a reciprocal only when it satisfies one geometrical condition.

A deformable figure admits a reciprocal only when it satisfies one more geometrical condition than the number of lines required to render it strictly indeformable.

A figure containing n surplus lines admits of $n - 1$ series of reciprocals, each series infinite.

The above may be expressed as follows:—

If v is the number of vertices in a figure, p the number of closed polygons into which it can be decomposed, the figure admits (*a*) *one* reciprocal; (*b*) an *infinity* of reciprocals; (*c*) *none*, according as v is (*a*) *equal* to; (*b*) *greater* than; (*c*) *less* than p.

The figures with which it is required to deal in practice are for the most part strictly indeformable.

27. **Examples of Reciprocal Figures.** — Fig. 35 consists of two five-sided polygons A ... E, *a* ... *e*, whose vertices A .... *a*, ... are joined by the lines A *a*, B *b* .... The total number of sides (15) is less than $2v - 3$ $(= 17)$, the figure is therefore deformable. If, however (as in Fig. 37), the figure is formed of two triangles A B C, *a b c*, whose angles are joined, then $s = 9 = 2 \times 6 - 3$, and the figure is strictly indeformable. Hence a figure of the form shown in Fig. 35 is deformable only so long as the number of sides in *each* of the two polygons A ... E, *a* ... *e* is *greater* than 3.

Moreover (§ 26), Fig. 35 does not admit a reciprocal unless it satisfies three geometrical conditions, i. e. one more condition than the number of additional sides required to render it strictly indeformable.

Similarly Fig. 36 does not admit a reciprocal unless it satisfies *one* geometrical condition.

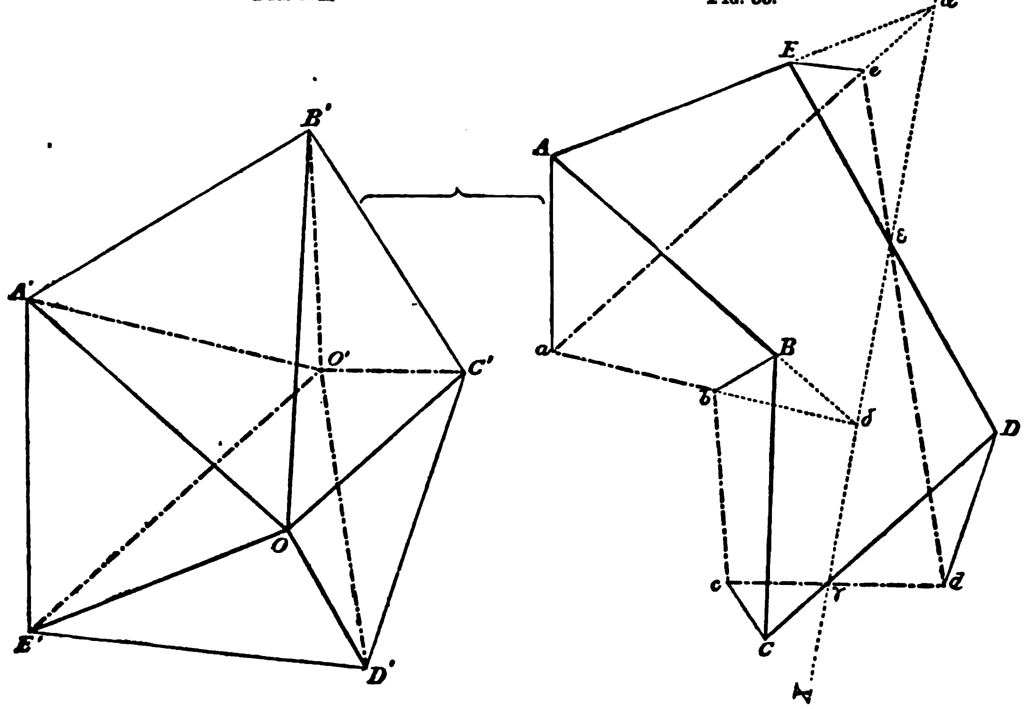
On the other hand any figure formed by joining the angles of any closed polygon to any point in its plane, admits of one reciprocal.

For example, to draw the reciprocal of Fig. 35. From any point O, Fig. 35A, draw vectors O A′, O B′, &c., respectively parallel to the sides A B, B C, &c., of the polygon A ... E (Fig. 35). Now to every vertex A, B ... (Fig. 35) there is a

F

corresponding triangle in Fig. 35A, and in every such triangle two of the sides will be the already drawn vectors from O. Starting from A′ draw A′ B′ parallel to B *b*, B′ *c*′ parallel to C *c*, C′ D′ parallel to D *d*, and D′ E′ parallel to E *e*. In order to obtain the reciprocal of the vertex A, E′ A′ must be drawn parallel to A *a*, but E′ and A′ are already fixed, and hence it is evident that if the polygon A . . . E (Fig. 35) is arbitrarily assumed in the first instance, it does not follow that the line joining E′ and A′ is parallel to A *a*, and if it is not parallel Fig. 35 does not admit of a reciprocal. Thus four out of the five lines A *a*, B *b* . . . . can be drawn arbitrarily, but the fifth is determined, since it must be parallel to a line passing through two points previously

FIG. 35A. FIG. 35.



fixed. In other words, as a first condition that Fig. 35 should admit a reciprocal, the sides A *a*, B *b* . . . . . . must be parallel to the sides of a closed polygon; this constitutes one geometrical condition. Suppose this condition fulfilled: it now remains to determine the reciprocals of the vertices *a* . . . *e* (Fig. 35) These reciprocals will all be triangles, and one side of *each* of these triangles is already determined. Hence, to draw the reciprocal of the vertex *a* (Fig. 35), draw A′ O′, E′ O′ in Fig. 35A, respectively parallel to *a b* and *a c*, thus fixing the position of a point O′. Passing to the vertex *b*, two sides A′ O,′ A′ B′ of its reciprocal triangle are already fixed, hence the third side O′ B′ must be parallel to $b\,c$, or Fig. 35 will not admit of a reciprocal. In fact any two sides $a\,b$, $a\,e$ of the polygon $a \dots e$ can be assumed arbitrarily, but the remaining three are then fixed. Hence the conditions that Fig. 35 should admit of a reciprocal are—1st, that the lines A a, B b .... should be drawn parallel to the sides of any closed polygon; 2nd, that the sides of the polygon A...E should be parallel to the vectors drawn to the angles of the polygon A′... E′ from some fixed point O in its plane; and, 3rd, that the sides of the polygon $a \dots e$ should be parallel to the vectors drawn to the angles of the polygon A′... E′ from some other fixed point O′ in its plane. In other words, the two polygons A ... E, $a \dots e$ must be funicular polygons of the lines A a, B b, &c., with respect to any two poles. But if this is the case, every pair of corresponding sides of the two funicular polygons must (§ 9) intersect on a straight line. It is easy to see that this involves only three geometrical conditions in the construction of Fig. 35. Draw any two pairs of corresponding sides at pleasure; e. g. A α A δ in the polygon A ... E and $a\,\alpha$, $a\,\delta$ in the polygon $a \dots e$, the intersections α and δ of these assumed pairs of sides determine a line X Y. Draw another side E D of the polygon A ... E at pleasure, the line E D cuts X Y in ϵ, then the side corresponding to E D in the polygon $a \dots d$ must be drawn through a fixed point ϵ. This constitutes one geometrical condition. Similarly assuming the two remaining

FIG. 36

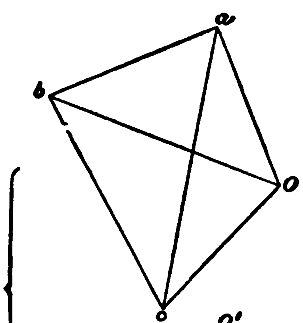


Fig. 36A

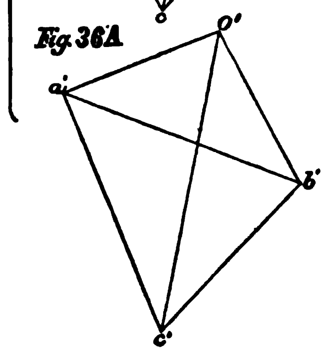


FIG. 37.

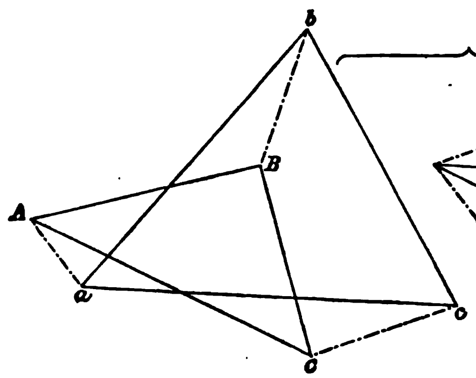
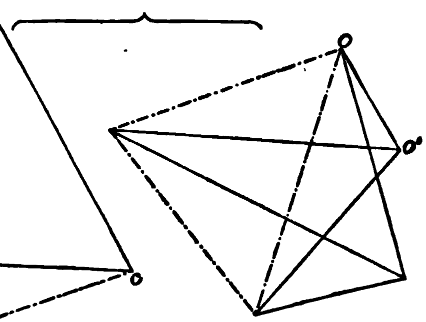


FIG. 37A.



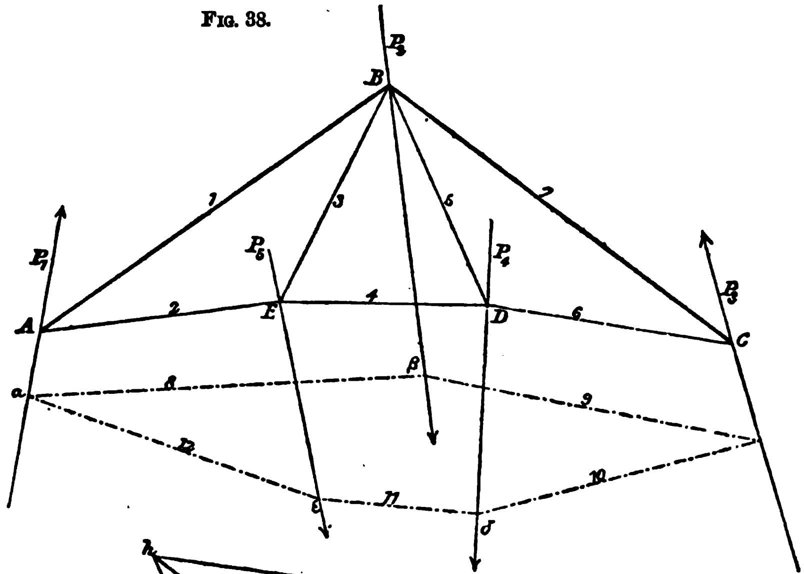
lines B C and C D, their corresponding lines $b\,c$, $c\,d$ must be drawn through fixed points γ, β respectively, thus making *three* geometrical conditions in all.

By similar reasoning it can be shown that, in order that Fig. 36 may admit of a reciprocal, *one* geometrical condition must be satisfied. Fig. 36A is the reciprocal of Fig. 36.

The construction of Fig. 37A, the reciprocal of Fig. 37, needs no explanation.

28. **Exceptional Figures.**—Fig. 38 is formed of a figure A . . . E, through the vertices of which the lines A α, B β, . . . are drawn, and the points α, β, . . . on these lines are joined, forming a polygon α . . . . ϵ. Call f the original figure A . . . E and F

FIG. 38.



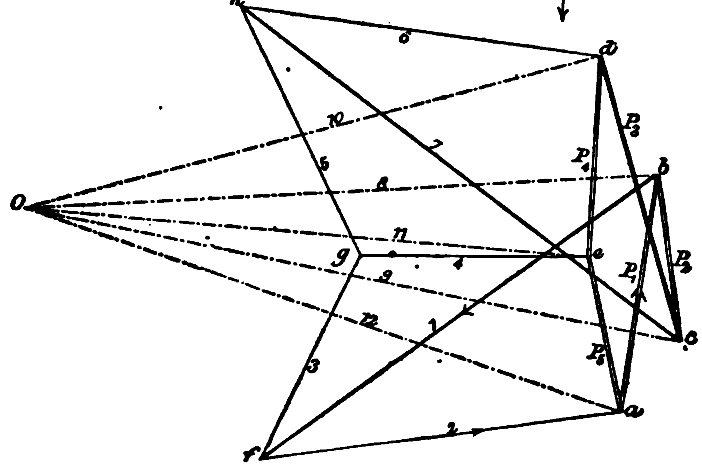


FIG. 38A.

the whole figure made up of f, the lines A α, B β . . . . and the polygon α . . . ϵ. Let s be the number of sides and v the number of vertices in the original figure. Then if

$$s = 2v - 3 + r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [a]$$

the figure f will (§ 25) be deformable in r different ways if r is negative, strictly indeformable if r is zero, and it will have r surplus lines if r is positive.

Suppose that through n of its vertices lines have been drawn, their extremities being joined by lines forming a closed polygon, then if s' is the number of sides in the complete figure F, and v' the number of its vertices,

$$s' = s + 2n \text{ and } v' = v + n.$$

Hence from equation [a]—

$$s + 2n = 2(v + n) - 3 + r$$

or,

$$s' = 2v' - 3 + r$$

Thus the complete figure F will be deformable, strictly indeformable, or will have surplus lines, according as the original figure f is deformable, indeformable, or has surplus lines.

Now, in Fig. 38, reciprocals of the vertices $\alpha \ldots \epsilon$ must be triangles, and if these triangles are drawn commencing with the vertex α, the last triangle would have one side determined by the first triangle, and another side determined by the last triangle but one, its third side would therefore be fixed, and if this third side were not parallel to the corresponding side of the original figure the latter would not admit of a reciprocal.

Hence, in order that a figure of the form shown in Fig. 38 may admit of a reciprocal, there is a condition with respect to the drawing of the lines A α, B β .... and $\alpha\beta$, $\beta\gamma$ ..., which is *perfectly independent* of the form of the original figure A ... E. Thus if the original figure A ... E had one surplus line, it should (§ 26) be possible to construct a reciprocal to the complete figure; but this will evidently be impossible unless the new condition above indicated is satisfied.

To satisfy this condition it is necessary that the polygon $\alpha \ldots \epsilon$ should be the funicular polygon of the lines A α, B β, ... Thus if the lines A α, B β are regarded as forces, the forces must either be in equilibrium, or one of them must be the resultant of the others.

If this condition is fulfilled, a figure of the form of Fig. 38 *will admit one reciprocal if the original figure A ... E is strictly indeformable; none if it has any surplus lines; while if it is deformable in r ways, it will not admit a reciprocal unless it satisfies r new conditions.*

The above is of importance in passing from the conception of purely geometrical figures to that of jointed polygonal frames with forces acting at their vertices, or joints.

[There are several other known exceptions to the rule stated in § 26, and new ones doubtless remain to be discovered: the exception above noticed alone occurs in the frameworks used in roof and bridge construction.]

CHAPTER IV.

STRESS DIAGRAMS.

29. **Mechanical Property of Reciprocal Figures.**—Suppose *a b c o* (Fig. 36) to be a polygonal frame formed of six bars jointed at the vertices *a*, *b*, *c*, *o*, and that at the extremities of each bar two equal and opposite forces are applied acting along the axis of that bar. Each pair of equal and opposite forces will be separately in equilibrium, and the whole frame will therefore be in equilibrium, whatever are the magnitudes of the equal and opposite pairs of forces. If the frame is in equilibrium, each separate vertex or joint must be in equilibrium, and the forces acting at that joint must therefore (§ 7) form a closed polygon. Hence there must be a figure such that to every vertex of the original figure *a b c o* there corresponds a triangle; but if such a figure exists, it will (§ 24) be the reciprocal of *a b c o*. Hence the problem of determining the stresses in the bars of such a figure resolves itself into the drawing of a reciprocal to the figure, and is possible or not in accordance with the conditions of possible reciprocity indicated in the preceding chapter.

The present figure has one surplus line, since $s = 2v - 3 + 1$, and $a' b' c' o'$ (Fig. 36A) is its reciprocal. The sides of the new figure will be proportional to the stresses in the corresponding bars of the given figure, and conversely a system of pairs of equal forces severally proportional to the sides of the figure $a' b' c' o'$, and applied to the extremities of the corresponding bars of the original figure *a b c o*, would maintain the latter in equilibrium.

30. **Determination of the Stresses in the Bars of a Polygonal Frame.**—A B C D E (Fig. 38) is a polygonal frame acted upon by five exterior forces P_1, P_2, P_3, P_4, P_5 applied at its several joints, and maintaining it in equilibrium. Whatever is the form of the frame, the exterior forces must satisfy the graphic conditions of equilibrium stated in § 8; that is to say, the polygon of forces and all funicular polygons of the system must close. Draw one of the funicular polygons $\alpha \beta \gamma \delta \epsilon$ of the system corresponding to any pole O (Fig. 38A), and suppose that at every vertex of this funicular polygon a force equal and opposite to the exterior force passing through that vertex is applied. Then each vertex of the funicular polygon will be separately in equilibrium, and hence to every vertex of the complete figure (Fig. 38) made up of the original frame and the funicular polygon, there must correspond a closed polygon in Fig. 38A. Thus, if three lines meet at any vertex in Fig. 38, there must be a corresponding triangle in Fig. 38A; if four lines meet at any vertex in Fig. 38, there must be a corresponding quadrilateral in Fig. 38A, and so on. Of two

figures thus related, each is the reciprocal of the other, and the reciprocal of the figures made up of the bars of a polygonal frame and the lines of external forces acting at its joints is termed the *stress diagram* of that frame.

The sides of the stress diagram are proportional to the exterior forces and to the stresses or interior forces in the corresponding bars of the frame.

Thus the conception of a jointed polygonal frame acted upon by exterior forces which produce stresses in its several bars is reducible to that of a purely geometrical linear figure conforming to the three conditions of § 25, and the statical problem of determining the stresses in the bars of a framework becomes the geometrical problem of determining the reciprocal of a given figure, and is possible, impossible, or indeterminate, according as the construction of the reciprocal is possible, impossible, or indeterminate.

Starting with the necessary hypothesis that the exterior forces $P_1 \ldots P_5$ (Fig. 38) maintain the given frame A ... E in equilibrium, then the complete figure admits of *one reciprocal* if the original figure is *strictly indeformable*, of an *infinite number of reciprocals* if the figure *has surplus lines*, while if the figure required *n new lines to render it strictly indeformable*, it would not admit a reciprocal unless it satisfied *n geometrical conditions*.

The problem of determining the stresses in the bars of a given framework is therefore, strictly speaking, determinate only when the latter is strictly indeformable, and this is the case with most of the frameworks with which it is necessary to deal in practice. If the framework has surplus bars the distribution of stress is indeterminate, and in dealing with such a framework it will be necessary to assume that one of the possible modes of distribution is the actual one. On the other hand, if the framework is deformable, it must have some special form; that is to say, there must be some special relations existing among the lengths of its bars, or the determination of the stresses is impossible.

To proceed with the drawing of the reciprocal in the present case. The reciprocals of the vertices $a \ldots \epsilon$ are already obtained, since they are the triangles, each made up of one of the sides of the polygon of forces and of the vectors drawn to O from its extremities. To obtain the reciprocal of the vertex A, draw a line from b parallel to B A, and from a parallel to E A. These lines intersect in f, then $a\,b\,f$ is the reciprocal of the vertex A. Passing to E, draw $e\,g$ parallel to E D and $f\,g$ parallel to E B, thus $a\,e\,g\,f$ is the reciprocal of the vertex E. For the vertex B, draw $c\,h$ parallel to C B and $g\,h$ parallel to D B, thus the five-sided figure $b\,c\,h\,g\,f$ is the reciprocal of B.

For the vertex D, join $d\,h$; then $d\,h$ should be parallel to C D, and $d\,e\,g\,h$ is the reciprocal of D. The reciprocal of C is $c\,d\,h$, all the lines of which have been already drawn.

It follows from the principle of the triangle and polygon of forces (§ 7) that the lines of the reciprocal figure give the stresses in the bars of the framework on the same scale as that adopted in drawing the polygon of forces. The lines of the reciprocal are figured similarly to the bars to which they correspond. The sides of the funicular polygon are similarly treated.

The following points in connection with the above should be noticed:—

1st. The polygon of forces or line of loads must first be drawn, and should close properly. In drawing this polygon, the forces should be taken one after the other right round the figure, as in Fig. 38.

2nd. Having drawn the reciprocal of any vertex, it is always necessary to proceed next to that one of the adjacent vertices at which the least number of unknown forces meet. Thus the vertex E, and not B, is dealt with after A. If this rule is observed, there will never be more than two unknown forces at any vertex, provided that the framework is strictly indeformable. If, however, the framework had surplus lines, there would be more than two unknown forces acting at some of the vertices, and the distribution of the stresses at those vertices would be (§ 7), strictly speaking, indeterminate.

3rd. It is evident that if the five forces $P_1 \ldots P_5$ are given, and if they are known to form a system in equilibrium, the reciprocals of the vertices A ... E can be drawn, and the required stresses obtained without the use of the funicular polygon. It should be remembered, however, that the figure thus obtained is *not* the reciprocal of the framework, but is only a portion of the reciprocal of the complete figure made up of the framework, the lines of action of the forces, and a funicular polygon.

4th. On arriving at the vertex D, the only unknown force is 6, and the line joining *h* and *d* (points previously obtained) must be parallel to 6. This closing up of the reciprocal figure serves as a verification of the accuracy of the preceding construction.

[NOTE.—In future the skeleton drawing of a framework will be called the "frame diagram," and the reciprocal the "stress diagram."]

31. **Distinction between Ties and Struts.**—From the stress diagram it can always be ascertained whether the stress in any bar of a framework is a tension or a pressure; i. e. whether the bar is a tie or a strut. Thus in the stress diagram (Fig. 38A) the triangle *a b f* represents the three forces acting at the joint A, and these forces being in equilibrium, the arrows point the same way round the triangle (§ 7). The direction arrow of P_1 is known, and this therefore fixes the other direction arrows. Transferring these arrows to the corresponding bars of the frame diagram, it will be seen that the direction arrow of 1 points *towards* the joint A, and that of 2 points *away* from it. Hence A B is in pressure, and A E in tension.

In the figures following, pressure bars are shown in strong lines, tension bars in fine lines.

32. **Roof Trusses with Symmetrical Vertical Loading.** Figs. 39, 40, Pl. I.; 41, 42, Pl. II.—In the preceding case all the exterior forces were supposed to be given; in practice it is required to deal with frameworks having known loads and resting on supports, the reactions of which must be treated as exterior forces forming with the known loads a system in equilibrium. In all stress diagrams, therefore, the reactions of the supports must first be determined, and then the polygon of the exterior forces (including those reactions) should be drawn.

When the loading is vertical and symmetrical, the reactions will be vertical, and each reaction will be equal to half the total load. Moreover the polygon of forces becomes a straight line, the "line of loads."

Fig. 39 is the frame diagram of a roof truss largely used for moderate spans up to about 60 feet. It consists of two rafters A C, C E, a polygonal tie A F G E, two struts B F, D G, and two braces C F, C G. The five upper joints A B C D E are taken as the loading points, the purlins carrying the adjacent bays of roof being supposed to pass over the joints B and D. If R_1 and R_2 are the respective reactions at A and B, then

$$R_1 = R_2 = \frac{2w_1 + 2w_2 + w_3}{2}.$$

Starting from a on the line of loads (Fig. 39a), set off $ac = w_1, cd = w_2, de = w_3, ef = w_2, fb = w_1$, successively from a to b, reading off their magnitudes in pounds or tons from any convenient scale. Bisect ab in o, then ao and ob represent the equal reactions R_1 and R_2.

From c and o draw cg and og respectively parallel to B A and F A, then the figure $acgoa$ is the reciprocal of the joint A, and cg and og give the stresses in the bars 1 and 2.

Proceeding next to the joint B (at which joint there are only two unknown forces), draw gh and dh respectively parallel to B F and C B. Then the figure $cdhg$ is the reciprocal of the joint B, and gh and dh give the stresses in the bars B F and B C.

Proceeding to the joint F, draw ht and ot parallel to F C and F G respectively, then the figure $othgo$ is the reciprocal of the joint F, and ht and ot give the stresses in the bars F C and F G.

Proceeding to the joint C, draw ti and ei parallel to G C and D C, then the figure $dhtied$ is the reciprocal of the joint C, and the lines ti and ei give the stresses in the bars G C and D C.

The whole of the stresses are now obtained, since by the symmetry of the structure and of its load, the stress in D G is equal to that in B F, and so on. It will be best, however, to complete the figure, proceeding next to the joint D and then to G. The figure $tikot$ is the reciprocal of the joint G, and in drawing this reciprocal the point k is fixed. If now the previous construction has been correctly carried out, the line joining k and f should be parallel to D E. By § 31 it can be ascertained which of the bars composing the frame diagram are in pressure and which in tension. In Fig. 39 and the succeeding figures the corresponding lines of the stress and frame diagrams are similarly numbered.

Fig. 39B is the stress diagram of the roof shown in Fig. 39, the loading being the same as before, with the addition of two equal loads w_4, w_4 suspended from the joints F and G. Each reaction is now equal to $w_1 + w_2 + w_4 + \frac{w_3}{2}$.

Starting from a, draw the polygon of the exterior forces (or load line), taking these forces in the following order: w_1, w_2, w_3, w_2, w_1, R_2, w_4, w_4, R_1. The reactions R_1 and R_2 will overlap each other on the load line as shown, and the whole length $a\,b$ of the load line will be unaltered by the addition of the two forces w_4.

The forces acting at the joint A are w_1, 1, 2, and R. From c and s on the load line draw lines intersecting in g, and respectively parallel to B A and F A. Then the five-sided figure $a\,c\,g\,s\,a$ is the reciprocal of the joint A, and the lengths $c\,g$, $s\,g$ represent the stresses in the bars A B, F A. The completion of the stress diagram needs no explanation.

Figs. 40, Pl. I., and 41, Pl. II., are the frame diagrams representing two forms of truss frequently used for spans of from 60 to 100 feet. Figs. 40a, Pl. I., and 41a, Pl. II., are the stress diagrams corresponding to the loading indicated. The drawing of these stress diagrams should present no difficulty.

Fig. 42, Pl. II., represents a modification of the truss shown in Fig. 39, which has been adopted for roofs up to 90 feet span, but which Professor Unwin recommends should be restricted to spans not exceeding 60 feet. In drawing the stress diagram (Fig. 42a, Pl. II.) a difficulty presents itself in dealing with the joint D. If the joints are taken in the order A, B, C, D, then on arriving at D there are three known forces w_3, 4, 5, and *three* unknown forces 7, 9, 8; the distribution of stress in the three bars 7, 9, 8 is therefore, strictly speaking, indeterminate. In fact it will be necessary to assume that the stresses 5 and 9 are equal,* so that 7 is equal to the resultant of the

* The assumption that the stresses 5 and 9 (Fig. 42, Pl. II.) may perhaps be deemed hardly satisfactory. A direct solution may be obtained without difficulty by the "method of sections." If a section be imagined by a plane perpendicular to the plane of the truss cutting not more than 3 bars, the stresses on which are unknown, the resultant of the external forces acting on one side of this section plane can be resolved (§ 10) in the 3 directions of these bars, and the resolved parts so obtained are the stresses in the respective bars. In the figure a vertical section plane may be taken a little to the left of the ridge cutting the bars 12, 13, and 14, and the total external force $w_1 + w_2 + w_3 + w_4 - R_1$ resolved along the 3 directions 12, 13, and 14, so determining the stresses in those bars. The stresses 8 and 11 at the junction of 8, 11, 12, and w_4 can then be determined in the usual way, and when 8 is known (as also the stresses in the bars meeting at B and C) we shall have only *two* unknown stresses, viz. 7 and 9, acting at D. If a section plane is imagined passing a little to the right of D, and cutting the bars 6, 7, 8, and 9, the stress acting in 6 (determined from the joint C) would have to be compounded with w_1, w_2, w_3, and R_1 as the external force acting on one side of the section plane.

equal tensions 5 and 9, and of the component of w_3 resolved along D C. In the stress diagram Fig. 42a, *d e* represents w_3, and *e f* drawn at right angles to *d* 9 gives the magnitude of the component of w_3 resolved in the direction of the bar D E. Draw *g i* parallel to the bar 9, and make *g i* equal to *g h*, since *g h* represents the stress 5. Then *h i* is the resultant of the two equal stresses 5 and 9; produce *h i* to *l*, making *i l* equal to *e f*, the whole line *h l* will therefore represent the resultant of 5 and 9 added to the component of w_3, hence *h l* gives the magnitude of the stress 7 in the bar D E.

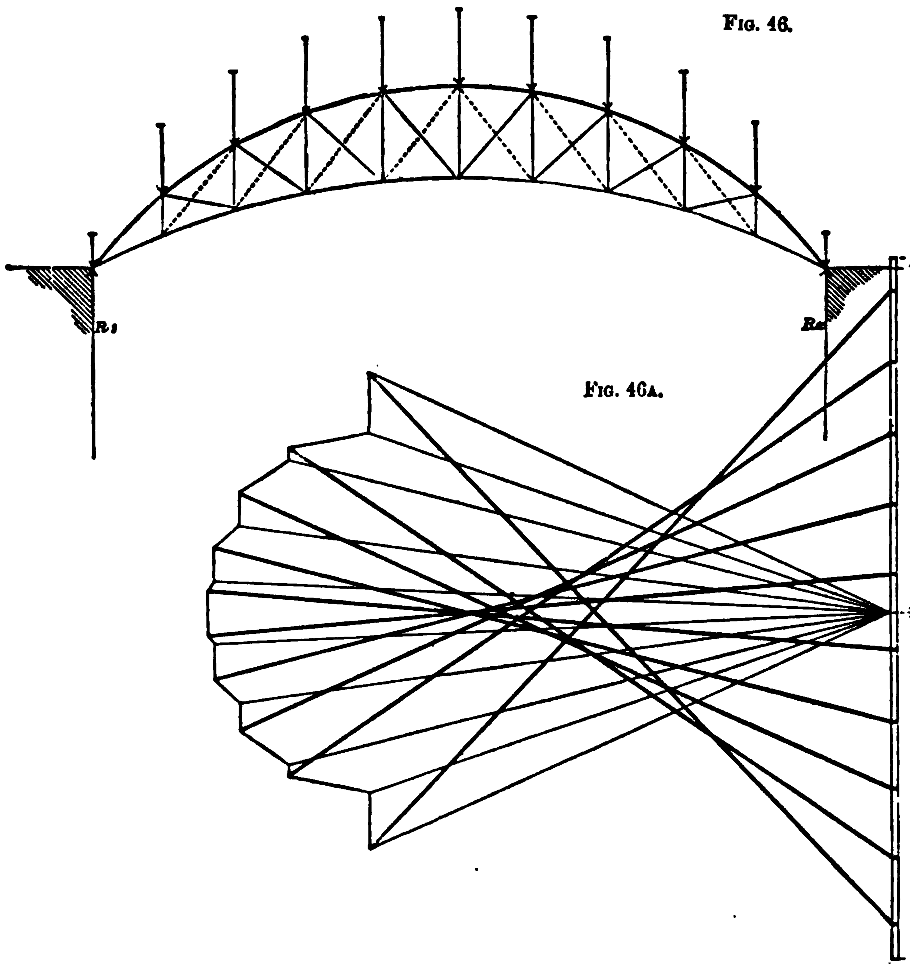


Fig. 46.

Fig. 46A.

Proceeding to the joint E, the four forces acting at this joint are 6, 7, 10, 14: draw *l r* parallel to 10 and *o r* parallel to 14, then the figure *o h l r* is the reciprocal of the joint E. The remaining construction will require no explanation. In Fig. 42a, Pl. II., only half the stress diagram has been drawn, the remaining half will be symmetrical about the line *o r*.

Fig. 44, Pl. III., shows another modification of the truss in Fig. 39, Pl. I., which can also be used for spans up to about 60 feet. Fig. 44a is the stress diagram for a symmetrical vertical loading applied at the upper joints.

Fig. 46 shows a bowstring roof, the form used for large spans. In roofs of this form the verticals are usually constructed as struts, and the diagonals as ties. If the loading is symmetrical, the single system of diagonals shown in continuous lines will be all in tension. If, however, the loading is unsymmetrical, as in the case of wind pressure, some of these diagonals would be placed in pressure, which pressure would be taken up as a tension by the diagonals shown in dotted lines. It would clearly be impossible to draw a stress diagram for the complete roof with its crossed diagonals, as the distribution of stress at every joint, except the end joints, is indeterminate. Before proceeding to draw the stress diagram for a bowstring roof it is necessary therefore to reduce the frame diagram to a single system of triangulation. Moreover, if the diagonals are to be all tension bars, then in deciding which system of triangulation to deal with, any bar which the stress diagram shows to be in pressure must be omitted, those only which are proved to be in tension being retained.

Fig. 46A is the stress diagram for Fig. 46. It will be found that the segments of the *arched rib* are the only bars in pressure.

33. **Roof Trusses with Unsymmetrical Vertical Loading.**—If the load is unsymmetrical, the reactions of the supports will be unequal, and the stresses in bars similarly situated with respect to the centre line of the truss will be unequal also. Suppose that the truss (Fig. 39, Pl. I.) is loaded only by the loads w_1, w_2, w_3 applied at the joints A, B, C respectively. To determine the reaction of the supports, set off w_1, w_2, w_3 successively from a to b on the line of loads (Fig. 39c), take any point O as pole, and draw the funicular polygon $a\beta\gamma\delta$ with respect to O; from O draw the line O o parallel to the closing line $a\delta$ of the funicular polygon, then (§ 14) $a\,o$, $o\,b$ represent the magnitudes of the reactions R_1, R_2 at the supports A and E respectively. In drawing this funicular polygon it should be noticed that since the lines of action of w_1 and R_1 coincide, the side of the funicular polygon parallel to the ray O a vanishes. The reciprocal can now be drawn, and it will be found, as might have been expected, that the strut G D is unstrained, and that the stresses 8 and 11 are equal.

34. **Pent Roof. Vertical Load.**—In this roof (Fig. 43, Pl. II.) the exterior forces are the five loads $w_1 \dots w_5$, and the forces supplied by the wall at the points of support A, B. The latter may be resolved into two forces P, Q acting in the direction of the bars which are fixed in the wall, and two vertical reactions r_2 and r_1 acting at A and B respectively. Set off the loads successively along $a\,b$ (Fig. 43a), then $a\,b$ represents their resultant R in magnitude, and since the loading has been taken symmetrical the line of action of this resultant passes through e, the centre point of B C. Join e A, and

from *a* (Fig. 43a) draw *a c* horizontal to cut *b c* parallel to *e* A in *c*. From *c* and *b* draw lines respectively parallel to A B, A C, and intersecting in *d*. Then (§ 11) *a c, c d, d b* represent the components of R resolved along, B C, B A, A C. But r_2, the reaction at A, is equal to w_5. Make *d f* equal to w_5, then *a b d f c* is the complete polygon of the exterior forces. The drawing of the stress diagram (Fig. 43a) presents no difficulty.

Wind Pressure.

35. Wind Pressure.—The maximum pressure of the wind in England has been variously taken at 40 and 50 lbs. per square foot of surface perpendicular to its direction. The usual direction of the wind is probably nearly horizontal, though it is possible that, in occasional gusts, this direction may make a considerable angle with the horizontal, becoming in fact normal to roof surfaces of high pitch.

For determining P_n, the normal pressure of the wind on any plane surface in terms of P, the pressure on a plane surface perpendicular to its direction, and *i* the angle of inclination of that direction to the plane of the surface, the following formula was deduced by Hutton from experiment:—

$$P_n = P \sin. i^{1 \cdot 84 \cos. i - 1}.$$

Supposing the direction of the wind to be horizontal, *i* is equal to the pitch of the roof surface.

The following table gives the value of P_n in lbs. per square foot of surface for horizontal winds acting with forces of 40 and 50 lbs. per square foot of vertical surface exposed to them, on roofs of various pitch:—

Angle of Roof.	P = 40.	P = 50.
5°	5·0	6·3
10°	9·7	12·1
20°	18·1	22·6
30°	26·4	33·0
40°	33·3	41·6
50°	38·1	47·6
60°	40·0	50·0
70°	41·0	51·3
80°	40·4	50·5
90°	40·0	50·0

In order therefore to determine the total stresses in the bars of a roof truss, it will be necessary—

1st. To draw a stress diagram for the dead or permanent load, i. e. the weight of the rafters, purlins, and roof covering,* together with the weight of the maximum snow-

* The weights per unit of area of different kinds of roof covering are given in the Appendix.

fall. The latter may be taken in England at from 5 to 6 lbs. per square foot of the *plan* of the area covered. Inasmuch as the maximum wind pressure is never likely to act simultaneously with the maximum snowfall, it is probable that, in England at all events, it will be quite sufficient to provide for the former only.

2nd. Assuming the direction of the wind to be horizontal, to draw either (*a*) a single diagram, corresponding to the normal pressure taken from the above table, or (*b*) to resolve the normal pressure into a horizontal and vertical component, and to draw a stress diagram for each component separately. Now if the *direction* of the wind is supposed normal to the roof surface, it is clear that the stresses due to this normal wind can be read off from the diagram drawn for the normal pressure of a horizontal wind by merely varying the scale. Similarly a diagram giving the stresses due to the horizontal component of a horizontal wind can be used to give the stresses due to the horizontal component of a normal wind.

3rd. To tabulate the stresses due to the dead or permanent load and to the wind pressure, distinguishing between tensions and pressures; and, finally, to form a table of total stress, by taking the sum or difference of the stresses due to the dead load and wind pressure, according as the stresses are of the same or of opposite kinds.

36. **Stress Diagrams for Normal Wind Pressure.**—In Fig. 44, Pl. III., the rafter AD is 19·4′, and the pitch is 30°. Hence, if the trusses are 10′ 0″ apart, and each truss is supposed to support half the adjacent bay, the total normal wind pressure supported by AD is $19 \cdot 4 \times 10 \times 26 \cdot 4$ lbs. If the purlins carrying the roof are supposed to be supported at B and C, the wind pressure $p_1 = p_4 = \frac{1}{6} \times$ total wind pressure; and $p_2 = p_3 = \frac{1}{3} \times$ total wind pressure; whence $p_1 = 0 \cdot 38$ tons, and $p_2 = 0 \cdot 76$ tons. If the rafter AD is considered as a beam uniformly loaded, and supported at A, B, C, D, the distribution of the pressure at the latter points will be somewhat different.

It is necessary first to determine the reactions Q_1 Q_2 at A and A[1]. In Fig. 44b, Pl. III., set off the loads $p_1 = 0 \cdot 38$ tons; $p_2 = 0 \cdot 76$ tons; $p_3 = 0 \cdot 76$ tons; $p_4 = 0 \cdot 38$ tons, along the load line *a b*. Take any convenient pole O, and draw the funicular polygon of the forces p_1, p_2, p_3, p_4, with respect to O; then O *o* drawn parallel to the closing side $\alpha\beta$ of the funicular polygon determines the two reactions Q_1 Q_2. Draw *o c* parallel to the bar 1, and *d c* parallel to 2. Then the polygon *o b d c* is the reciprocal of the joint A, and *o c*, *d c* represent the stresses due to wind pressure in the bars 1 and 2 respectively. Complete the stress diagram, and it will be found that the bars 11 and 13 are unstrained, and consequently the stresses in 10, 12, and 14 are equal.

If, as is the case in all large iron roofs, one end of the truss rests on rollers, so as to allow of expansion due to variations of temperature, then, since only one abutment can furnish the horizontal component of the reaction, it will be evidently necessary to draw two diagrams for wind pressure, since the stresses in the bars forming the truss will be different when the wind acts on different sides of the roof.

Suppose, in Fig. 44, Pl. III., that the rollers are placed at the end A^1, then this abutment can furnish only the vertical component of its reaction, and the whole horizontal component of both the reactions Q_1 and Q_2 must be borne at A. In Fig. 44c, Pl. IV., draw the load line *a b*, and set off along it the four loads and the two reactions. Through *o* draw a horizontal line, and terminate it on perpendiculars dropped from *a* and *b*; then *c a* is the vertical component of Q_2 and *b d* of Q_1, while *c d* represents the sum of the horizontal components of Q_1 and Q_2. Now the forces acting at A are the load p_1, the vertical component of Q_1, the horizontal components of Q_1 and Q_2, and the stresses in the bars 1 and 2. From *c* draw *c e* parallel to the bar 1, and from *g* draw *g e* parallel to 2. Then the five-sided polygon *g b d c e* is the reciprocal of the joint A (Fig. 44, Pl. III.), and the lines *c e*, *e g* represent the stresses in the bars 1 and 2. Completing the stress diagram, the triangle *c a f* will be the reciprocal of the joint A^1, since at this joint there act only the bar stresses 14, 15, and the vertical component (*c a*) of Q_2. It will be seen that the bars 11 and 13 are unstrained, and hence the stresses in 10, 12, 14 are equal.

Now suppose the rollers to be at A, the wind coming from the same side. The polygon of forces *a b d c* (Fig. 44d, Pl. IV.) will be precisely the same as before, but the horizontal component *c d* will now act at A^1 instead of at A Hence at A there is only the load p_1, the vertical component (*b d*) of Q_1, and the stresses 1 and 2. Draw *d e*, *g e* parallel to 1 and 2 respectively, then the four-sided figure *g b d e* is the reciprocal of the joint A (Fig. 44), and the sides *d e*, *g e* represent the stresses in the bars 1 and 2 respectively. Completing the stress diagram, the triangle *c a d* is the reciprocal of the joint A^1, and the bars 8, 9, 11, 13, 15 are unstrained. These bars are, however, brought into play as soon as the stress in D A^1 ceases to be a simple pressure, and D A^1 begins to bulge.

37. Diagrams of Horizontal and Vertical Components of Wind Pressure.— To draw the horizontal component diagram, obtain first the horizontal components h_1, h_2, h_3, h_4 of the forces p_1, p_2, p_3, p_4 (Fig. 44, Pl. III.), then H (Fig. 45, Pl. IV.) equal to the sum of the horizontal components $h_1 + h_2 + h_3 + h_4$ will be the magnitude of their resultant, and by symmetry H will act at *s*, the middle point of the rafter A D. Suppose the truss fixed at the support A^1, and furnished with rollers at A, then in order that equilibrium may be maintained, the support A, though unable to furnish a horizontal reaction, must be supposed capable of supplying a vertical downwards-acting force *v* sufficient to prevent the truss from rotating about A^1. Since R, H, and this vertical force *v* are the only forces acting on the frame, their directions must (§ 8) pass through a point. Draw a vertical through A, cutting the line of action of H in *t*, then the line of action of R must pass through t. For the stress diagram (Fig. 45a) draw the triangle $b\,a\,c$ of the exterior forces, making $b\,a$ equal to H, then $a\,c$, $c\,a$ represent v and R respectively. Set off h_1, h_2, h_3, h_4 successively from a along $a\,b$. The four forces acting at A are h_1, v, and the stresses in 1 and 2. Draw $e\,d$, $c\,d$ parallel to 2 and 1, then $e\,d$, $c\,d$ represent the stresses 2 and 1 respectively. Proceed to the joint B, and complete the diagram; as before, the struts of the right half-truss are unstrained.

The construction of the stress diagram, when the truss is fixed at A and free to move at A^1, presents no difficulty. In this case the support A must furnish the oblique reaction R, while A^1 supplies a vertical *upward*-acting reaction equal to v.

The vertical component diagram can be drawn in the same way as Fig. 39c, Pl. I., since it is merely the case of unsymmetrical vertical loading. Only *one* vertical component diagram need be drawn, since the wind blowing from the other side will merely cause the stresses in symmetrically situated bars to be interchanged.

The stresses obtained from the dead load and wind pressure diagrams must now be tabulated and added with due regard to their *sign*. The maximum stress in each bar is thus ascertained.

38. **Bowstring Roof with Normal Wind Pressure.** Figs. 47, 47A, 47B.—The load at any joint due to wind can be taken as equal to the pressure which the wind would exert upon an area equal to the portion of roof surface supported by one bay of the rafter, and inclined at the same angle to the horizontal as the tangent to the curve of the rib at that joint. Supposing the direction of the wind to be horizontal, it will be necessary to take from the table, page 45, the normal pressure corresponding to the several inclinations of these tangents, and to multiply them by the area of roof surface supported by one bay of the rib, or in the case of the end joint by half this area. The pressures p_1, p_2, p_3, p_4, p_5 (Fig. 47) are thus obtained, and their resultant R will, if the curve of a rib is a circle, pass through its centre C. If B is the roller end of the roof truss, the entire *horizontal* component of this resultant must be balanced by a horizontal reaction at A, while the *vertical* component of the reaction will be borne partly at A and partly at B. Draw the polygon of the forces $p_1 \ldots p_5$ (Fig. 47A), then the closing line $f\,a$ of this polygon gives the magnitude and direction of R. Through A and B draw lines parallel to the direction of R. Take any pole O, and resolve R by means of the funicular polygon α, β, γ (§ 13) into the two parallel components $a\,o$, $o\,f$ (Fig. 47A) acting at B and A respectively; draw a horizontal line $g\,h$ through o, terminated by perpendiculars from a and f. Then the forces acting at A will be p_1, the reaction H $(= g\,h)$, V $(= f\,h)$, and the stresses 1 and 2. From g and e draw $g\,k$, $e\,k$ parallel to 1 and 2 respectively; then

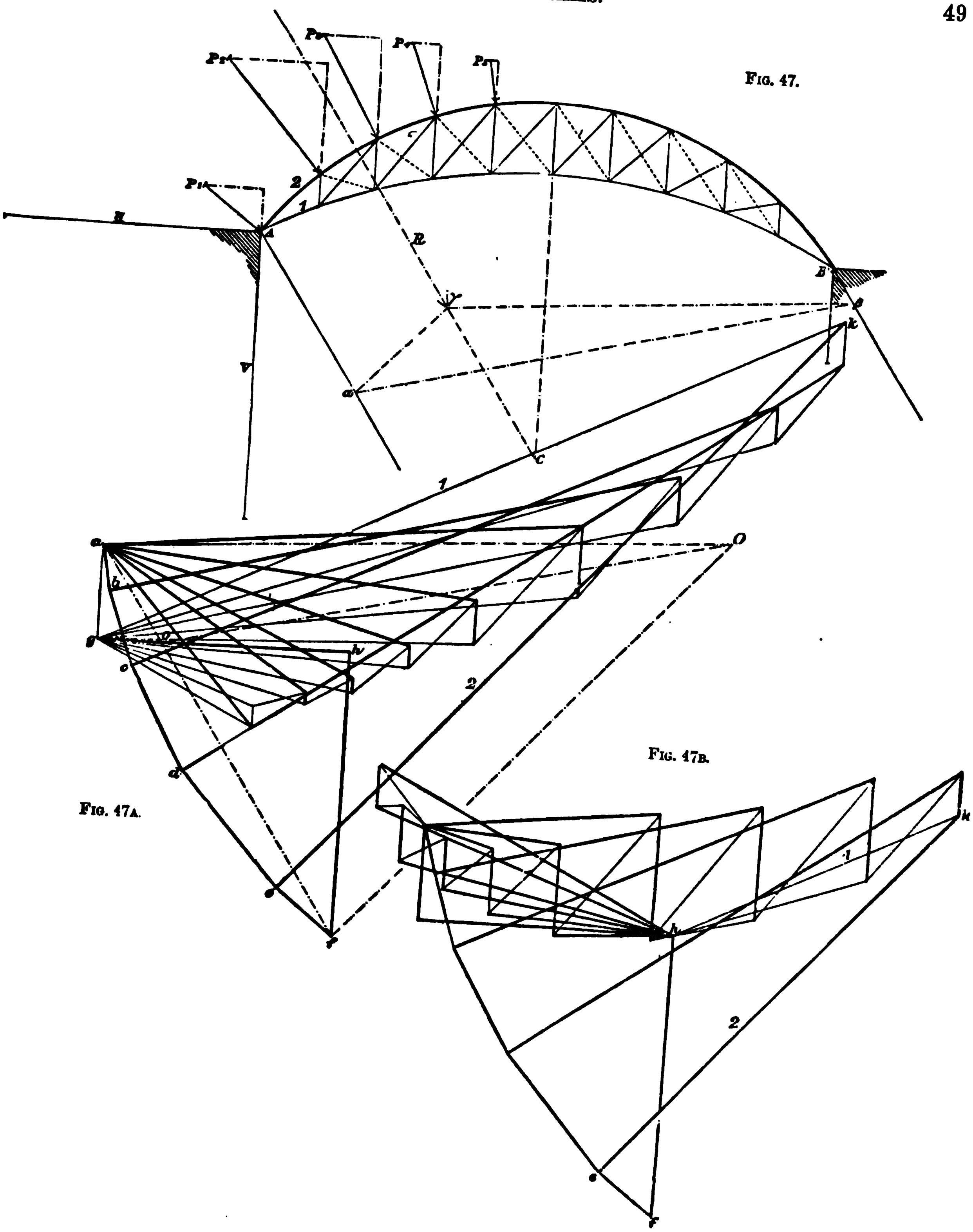


FIG. 47.

FIG. 47A.

FIG. 47B.

the figure $efhgk$ is the reciprocal of the joint A, and gk, ek represent the stresses in bars 1 and 2. Complete the stress diagram in the usual way. The diagonals are all in tension, the verticals are all in pressure except those nearest the springing on each side.

If A is the roller end of the roof truss, the horizontal component gh of the wind pressure must be borne entirely at B. The forces acting at A will be p_1, V, and the stresses 1 and 2; draw hk, ek (Fig. 47B) parallel to the bars 1 and 2, then the figure $efhk$ is the reciprocal of the joint A, and hk, ek represent the stresses in 1 and 2 respectively. Complete the diagram, and it will be seen that a very considerable alteration in the stresses has taken place. The segment of the tie nearest to B is in *pressure*, and the segment of the rib terminating at the same point in *tension*. If these abnormal stresses are not counterbalanced by the stresses due to dead load, the fact of their possible occurrence must be taken into consideration in the design of the roof.

39. **Warren Girder.**—Fig. 48 represents the frame diagram of a Warren girder loaded with equal loads at its upper joints. Fig. 48A represents half the stress

FIG. 48.

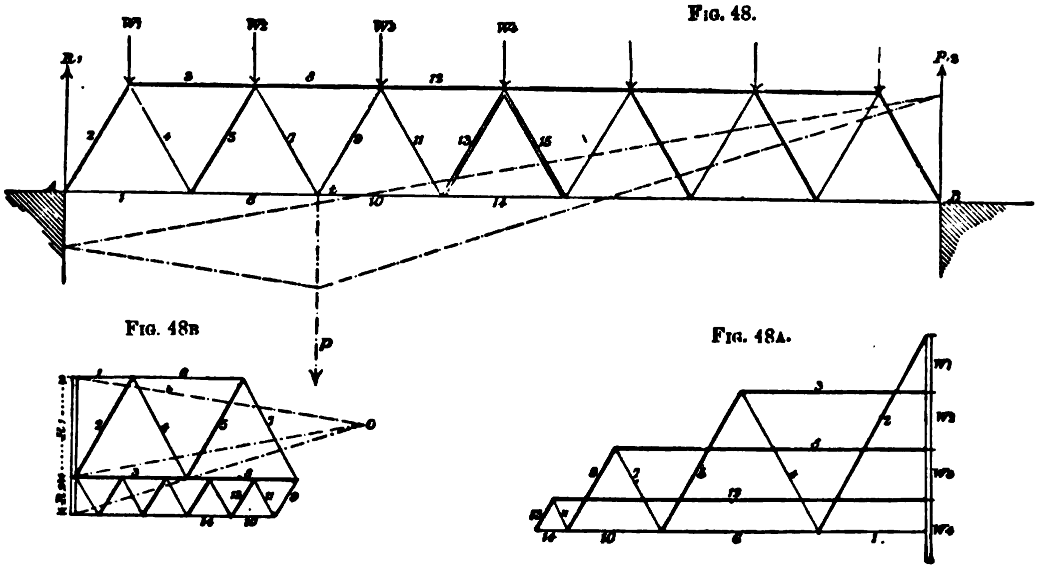


FIG. 48B

FIG. 48A.

diagram; the stresses in symmetrically situated bars are equal. Fig. 48B is the stress diagram for the same girder when loaded with a single load p at one of the lower joints t. The magnitudes R_1 and R_2 of the reactions of the left and right support are obtained by a funicular polygon in the usual way. It should be noticed that with this loading the diagonals 7 and 9 are both in tension, while the diagonals on either side are alternately in pressure and tension. Moreover, the stresses in all the diagonals to the right and left of the joint t are respectively equal, and these stresses are directly proportional to the reactions at B and A, and inversely proportional to B t, A t. The stresses in the segments of the boom increase in arithmetic series from the abutments to the loaded joint. If p is applied on the *left* half girder, 13 is a *tension* and 15 a *pressure*: the reverse is the case if p acts on the *right* half girder.

40. **Bollman Girder.**—In this girder (Fig. 49) the span is divided into equal bays by equally long verticals, the lower ends of each of which are tied back to the extremities of the boom. The stress diagram (Fig. 49A) of this girder is drawn for equal loads w_1, w_2, w_3 applied at the upper ends of the verticals. The drawing of

FIG. 49.

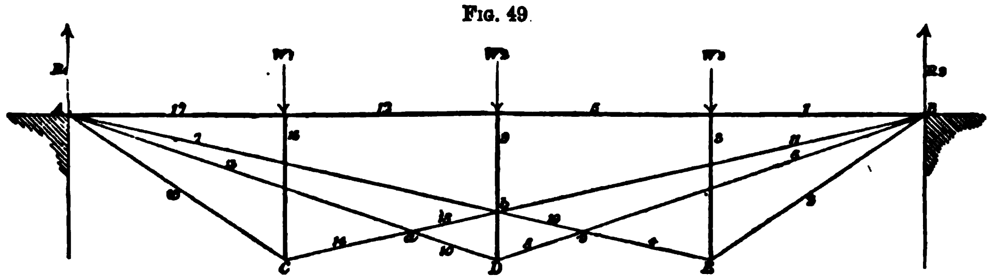
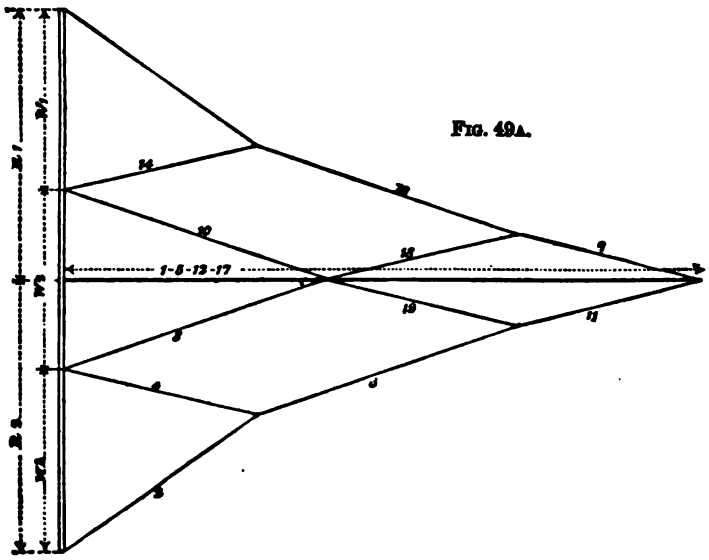


FIG. 49A.



this diagram brings out one peculiarity worthy of notice. For the purposes of the reciprocal the loads may be considered to be applied at the lower sides of the verticals, and the pressures 15, 9, 3 in the latter will be respectively equal to w_1, w_2, w_3. Now it will be found impossible to construct a stress diagram which closes properly unless the points a, b, c (Fig. 49), at which the ties cross, are considered as joints. Thus, for the purposes of the stress diagram, the tie C B must be considered as made up in three segments, figured 14, 18, 11, and A D in two lengths 13 and 10. In the stress diagram the reciprocals of the joints a, b, c form parallelograms, as shown in Fig. 49A. In the present case it would not, of course, be necessary to draw the complete stress diagram in order to obtain the stresses in all the bars of the frame diagram, Fig. 49. The reciprocal triangles of the joints C, D, E give the stresses in all the ties, and the stress in the boom is equal to the horizontal component of the resultant of the tensions of all ties meeting at either A or B. If, however, in such case as this it is desired to draw the complete stress diagram, it must be remembered that such intersections as *a*, *b*, *c* (Fig. 49) must be treated as joints, otherwise a diagram which will close up properly cannot be obtained.

The method of reciprocal diagrams is not in general adapted to girders subject to travelling load, as the construction of the number of diagrams required in investigating the stresses for different positions of the load would be a tedious operation. The method, however, is sometimes useful for obtaining the stresses in the *booms*, since the latter suffer their maximum stress when the girder is fully loaded, and it may often be employed with advantage in ascertaining the stresses due to dead load; i. e to the weight of the structure itself and of the roadway.

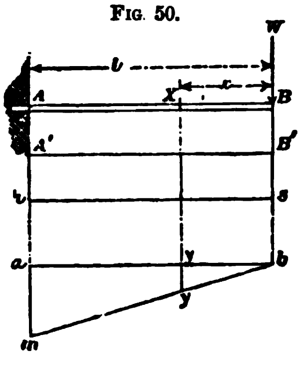
CHAPTER V.

ACTION OF STATIONARY LOADS.

Beams fixed at one end.

41. Single Concentrated Load.—By a beam is understood a single continuous bar, such that the axis, i. e. the line joining the centres of all transverse sections, is straight; and further, that there is a plane of symmetry containing the axis. The loads are supposed to act in this plane of symmetry with directions normal to the axis. Such a bar A B (Fig. 50) is fixed at A, and supports a load W at the free end B. Considering any section X, suppose two opposite forces S′ (acting up) and S (acting down) equal and parallel to W to be applied at X. Of these forces W and S′ form a couple whose arm is x, and S is unbalanced. The moment W . x of the couple is termed the *bending moment* at the section X, and is resisted by the moment of an opposite stress couple, termed the *moment of resistance*, acting in the interior of the beam. The force S is termed the *shearing force* at the section X, and is met by the equal and opposite *shearing stress*, or resistance of the section to shearing.

FIG. 50.



Shearing Stress.—The shearing stress at any section of a loaded beam is therefore equal and opposite to the resultant of all the exterior forces acting on one side of that section. The shearing stresses at all sections of the beam A B (Fig. 50) are equal to W. Hence the distribution of shearing stress will be graphically represented by a rectangle A′ B′ *h s*, of which the length is equal to that of the beam and the height is equal to W, set off from any convenient scale. The action of S′ (= W) at any section X tends to cause the part X B to the right of that section to move down relatively to the remaining portion X A of the beam, the shearing stress S, or molecular resistance to shearing at X, must therefore act upwards. *Shearing stress at any section which resists a tendency of the portion of the beam to the right of that section to move down under the action of the exterior forces will be termed negative.* Thus if the load W were exchanged for an upward thrust, the shearing stress at any section of the beam A B would be termed positive. Figs. 51 and 51A show these two cases. Shearing is supposed to

FIG. 51.

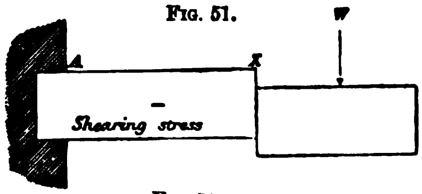


FIG. 51A.

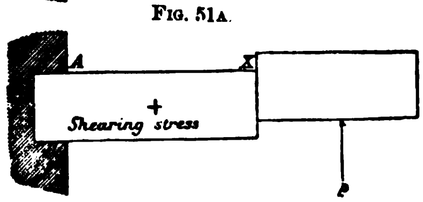


FIG. 52.

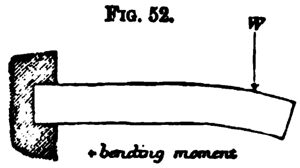
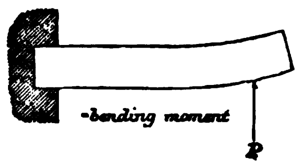


FIG. 52A.

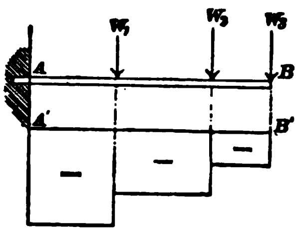


have begun to take place at the section X. In Fig. 51 the shearing stress from A to X is *negative:* in Fig. 51A it is *positive.* If W acts at any point between A and B (Fig. 50), the shearing stress at any section to the right of that point is *nil* if the weight of the beam itself is neglected; while for sections to the left it is negative, and equal to W. In accordance with the above convention with regard to the signs of shearing stresses, loads will be considered as positive forces, upward-acting reactions as negative.

Bending Moments.—The bending moment at any section of a beam is equal to the moment of all the exterior forces acting on one side of that section. The bending moment at X (Fig. 50) is equal to $W \,.\, x$ and becomes zero, and $W \,.\, l$ when $x = o$ and l respectively. Draw the right-angled triangle $a\,b\,m$, making $a\,b$ equal to l, and $a\,m$ equal to $W \,.\, l$, then the ordinate $(y\,y)$ of this triangle under any section X gives the bending moment at X on the same scale as that adopted in setting off $a\,m$. The triangle has been drawn right-angled, but this is obviously non-essential, any triangle which has its base equal to $a\,m$ and height equal to l will equally represent the bending moments. In accordance with the convention adopted in § 17, the bending moment for the case shown in Fig. 52 is termed *positive*, and for that shown in Fig. 52A *negative*.

42. **Several Concentrated Loads.**—*Shearing Stress.*—The beam A B (Fig. 53) supports those loads W_1, W_2, W_3. The shearing stress for any section lying between A and the line of action of W is $W_1 + W_2 + W_3$. Between W_1 and W_2 the shearing stress is $W_2 + W_3$, and between W_2 and W_3 it is W_3. Hence the distribution of shearing stress may be represented by a figure made up of rectangles whose breadths are equal to $W_1 + W_2 + W_3$; $W_2 + W_3$; and W_3; and whose lengths are equal to that of the segments into which the beam is divided by the points of application of the loads. The shearing stress is all *negative*. If the end of the beam projects beyond W_3, the shearing stress in the projecting portion will be zero, if the weight of the beam is left out of account.

FIG. 53.



Bending Moment.—The bending moment diagram for each separate load can be drawn as in the preceding paragraph, then the sum of the three ordinates under any section is the ordinate of the figure required. Another construction will be given later.

43. **Uniformly Distributed Load.**—*Shearing Stress.*—The beam A B (Fig. 54) supports a load w per unit of length. The shearing stress at any section X is negative, and equal to $w.x$, becoming zero at B, and $w.l$ at A. The distribution of shearing stress will therefore be represented by a triangle $s\,h\,e$, whose height $s\,h$ is equal to l, and base $s\,e$ is equal to $w.l$ set off from any convenient scale.

FIG. 54.

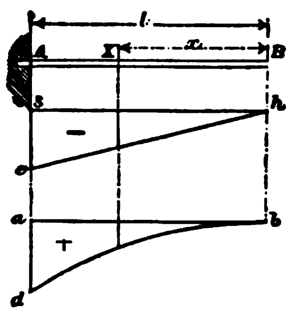
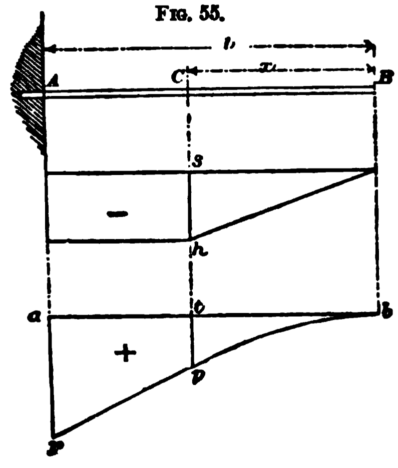


FIG. 55.



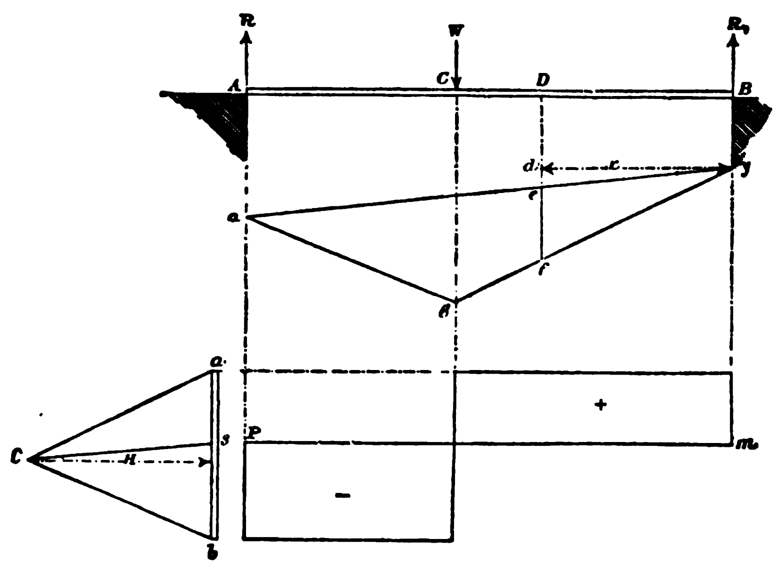
Bending Moment.—The bending moment at X is $w.x.\frac{x}{2}$, and becomes zero at B, and $\frac{w.l^2}{2}$ at A. The bending moment at any section is therefore given by the ordinate of a parabola $b\,d$ (Fig. 54) whose vertex is at b, and whose ordinate $a\,d$ at a is equal to $\frac{w.l^2}{2}$, and which touches $a\,b$ at b. If the scales are so arranged that $a\,d$ is not greater than $\frac{l}{4}$, a circular arc will sufficiently approximate to the curve.

If the beam A B (Fig. 55) is loaded with a uniformly distributed load from B to C only, the distribution of shearing stress is evidently represented by a rectangle under the segment A C, and a triangle under C B, the rectangle and triangle having a common side $s\,h$ equal to $w.x$. The bending moments under this condition of loading are given by the ordinates of a straight line under A C and a parabola under C B, the parabola and straight line having a common ordinate $t\,p$ equal to $\frac{w.x^2}{2}$, while the extreme ordinate $a\,r$ of the straight line is equal to $\frac{w.x}{2}(2\,l-x)$. The representation of the distribution of shearing stress and bending moment under other conditions of loading can be obtained by combining the results above obtained.

Beams supported at both ends.

44. **Single Concentrated Load.**—*Shearing Stress.*—The beam A B (Fig. 56) supports a single concentrated load W acting at C. The only exterior force acting to

FIG. 56



the right of the section C is $-R_1$, the reaction of the abutment B. Hence the shearing stress at D, or any other section between C and B, is *positive* and equal to R_1. Similarly the shearing stress from A to C is *negative* and equal to $-(W-R_1)$. (Taking moments about A; $R_1 . AB = W . AC$, hence the shearing stress from C to B is equal to $W . \frac{AC}{l}$, and from A to B is $- W . \frac{CB}{l}$.)

Draw the load line ab representing W on any convenient scale, and draw the funicular polygon $\alpha \beta \gamma$ with respect to any pole O; thus (§ 14) O s parallel to $\gamma \alpha$, the closing line of this polygon, divides ab into two parts as, sb, representing R_1 and R respectively.

The graphic representation of shearing stress will be two rectangles, one of which is vertically under the segment AC and the other under CB, the heights of these rectangles being respectively equal to sb and as.

It will be noticed that in the figure as drawn, a horizontal line pm drawn through s separates the positive and negative portions of the shearing stress diagram. The *positive* shearing stress is shown *above* the line pm, and the *negative below*.

Bending Moment.—In Fig. 56 the only exterior force acting to the right of the section D is R_1. Hence the bending moment at D is $R_1 . DB$; or, $R_1 . x$. Let fall a vertical from D, cutting the two sides $\alpha\gamma$, $\beta\gamma$ of the funicular polygon in e and f. Since the triangle $ef\gamma$ is similar to the triangle a O s,

$$\frac{ef}{as} = \frac{x}{H}; \text{ or, } ef . H = as . x = R_1 . x$$

where H is the polar distance (§ 15). Similarly the bending moment at any section is equal to the product of the ordinate (measured parallel to the load) of the funicular polygon under that section into the polar distance. The bending moment at any section of the beam is, therefore, proportional to the ordinate of the funicular polygon vertically below it; and, if the polar distance is taken as unity on the linear scale, these ordinates read off from the scale of loads will give the bending moments at all sections.

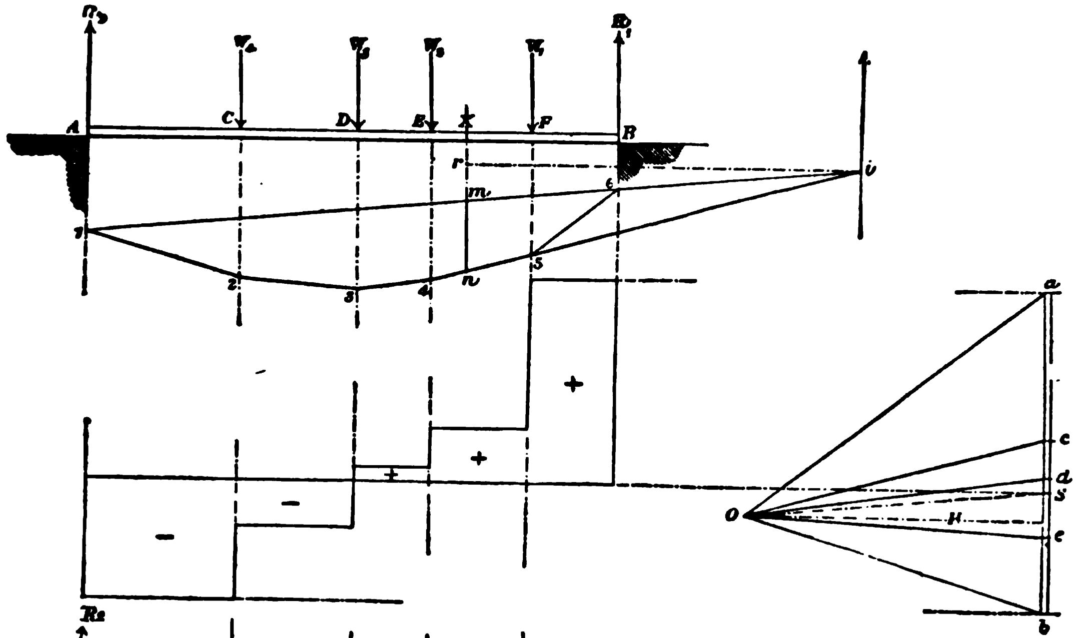
45. **Any Number of Concentrated Loads.**—*Shearing Stress.*—The beam AB (Fig. 57) supports four concentrated loads $W_1 .. W_4$ acting at F, E, D, C, respectively. On a load line ab set off $ac = W_1$; $cd = W_2$, and so on. Draw the funicular polygon 1....6 of the loads with respect to a pole O; then (§ 14) OS parallel to the closing line 16 of this polygon divides ab into aS, Sb, representing R_1 and R_2 respectively.

The shearing stress at X is equal to $R_1 - W_1 = aS - ac = Sc$. Similarly the shearing stress at any section between E and D is $R_1 - (W_1 + W_2) = Sd$, and generally the shearing stress for the several segments of the beam is given by the distances from S of the divisions on the load line.

The whole figure representing the distribution of shearing stress is made up of rectangles obtained by drawing vertical lines through the points of application of the loads, and horizontal lines through the divisions on the load line. The horizontal through S separates positive and negative shearing stress, and the change of sign takes place at the section D.

It follows from the above that, if the loading is symmetrical and symmetrically applied with respect to the centre of the beam, no load being applied *at* the centre, there will be a segment at the centre, in which the shearing stress is *nil*.

Fig. 57.



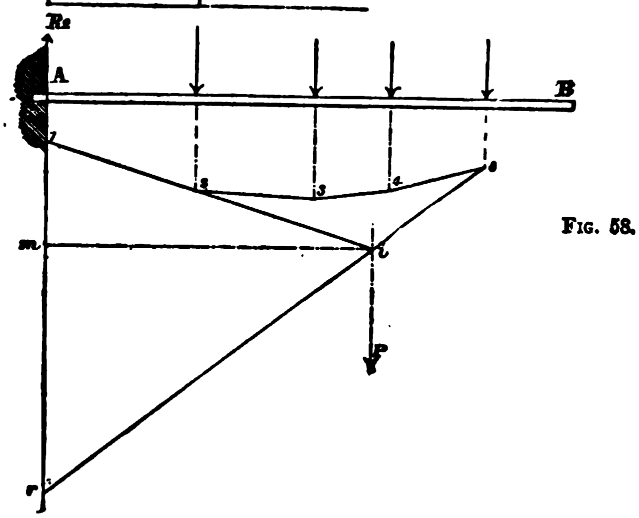


Fig. 58.

Bending Moments.—The bending moment at X is equal to the moment of the resultant of R_1 and W_1 about X. The magnitude of this resultant is $-R_1 + W_1 = -Sa + ac = -Sc$: it is therefore negative, and its line of action passes through *i*, the intersection of the sides 16, 45 of the funicular polygon produced (§ 12). The bending moment at X is therefore equal to $-Sc.ir$, when ir is the perpendicular from i on the vertical through X. The triangle $m\,n\,i$ is similar to S c O. Hence

$$\frac{m\,n}{S\,c} = \frac{i\,r}{H}; \text{ or } m\,n\,.\,H = S\,c\,.\,i\,r,$$

where H is the polar distance (§ 15).

Hence, as in the preceding section, the bending moment at X is equal to the product of the ordinate $m\,n$ of the funicular polygon under X, into the polar distance. If H has been taken as unity on the linear scale, then $m\,n$ read off from the scale of loads gives the bending moment at X, and similarly for all other sections of the beam. In Figs. 56 and 57, H, if not unity, should be taken at some convenient whole number from the linear scale, then the ordinates of the funicular polygon read off from a properly constructed scale (§ 17) will give the bending moments.

If the equal similarly loaded beam A B (Fig. 58) is fixed at A, the reaction R_2 at A will be equal to $W_1 + W_2 + W_3 + W_4$. Using the same pole and polygon of forces, the funicular polygon takes the form shown, and its two last sides intersect in i, a point on the line of action of the resultant P of the four loads (§ 12). Produce 5 i to meet the vertical through A in r, then the moment of P about A is P . $m\,i$, and, since the triangles 1 $i\,r$ and a O b are similar, P . $m\,i$=1 r . H. Hence 1 r represents the bending moment at A, and similarly the ordinate of the funicular polygon under any other section gives the value of the bending moment at that section.

46. **Uniformly Distributed Load.**—*Shearing Stress.*—The beam A B (Fig. 59) of length l is subjected to a load w per foot run, the whole load is therefore equal to $w\,.\,l$, and the reactions are each equal to $-\frac{w\,.\,l}{2}$. The exterior forces acting to the right of a section X, distant x from A are $-R_1$, and a load equal to $w(l-x)$. Hence the shearing force at X is

$$-\frac{w\,.\,l}{2} + w\,(l-x) = \frac{w}{2}\,(l-2\,x),$$

and the shearing stress is $-\frac{w}{2}\,(l - 2\,x)$.

This equation shows that the ordinates which represent the shearing stress at the several sections of the beam are those of a straight line. Putting $x = o$; or $x = l$, the shearing stresses at the abutments A and B are $-\frac{w.l}{2}$ and $+\frac{w.l}{2}$ respectively. Putting $x = \frac{l}{2}$, the shearing stress at the centre of the beam is 0.

Make $r\,s$, $r'\,s'$ each equal to $\frac{w\,.\,l}{2}$, and join $s\,s'$; then the whole figure $s\,r\,r'\,s'$ represents the distribution of shearing stress, and the ordinate of this figure under any section of the beam gives the shearing stress at that section. The shearing stress is *positive* for the right half of the beam, and *negative* for the left half.

Bending Moments.—The bending moment at X is equal to $- R_2 \,.\, x + w \,.\, x \,.\, \frac{x}{2}$

$$= -\frac{w}{2} \,.\, l \,.\, x + w \,.\, \frac{x}{2} = -\frac{w\,x}{2}(l-x).$$

The bending moments at the several sections of the beam A B are therefore represented by the ordinates of a parabola. The uniformly distributed load may in

FIG. 59.

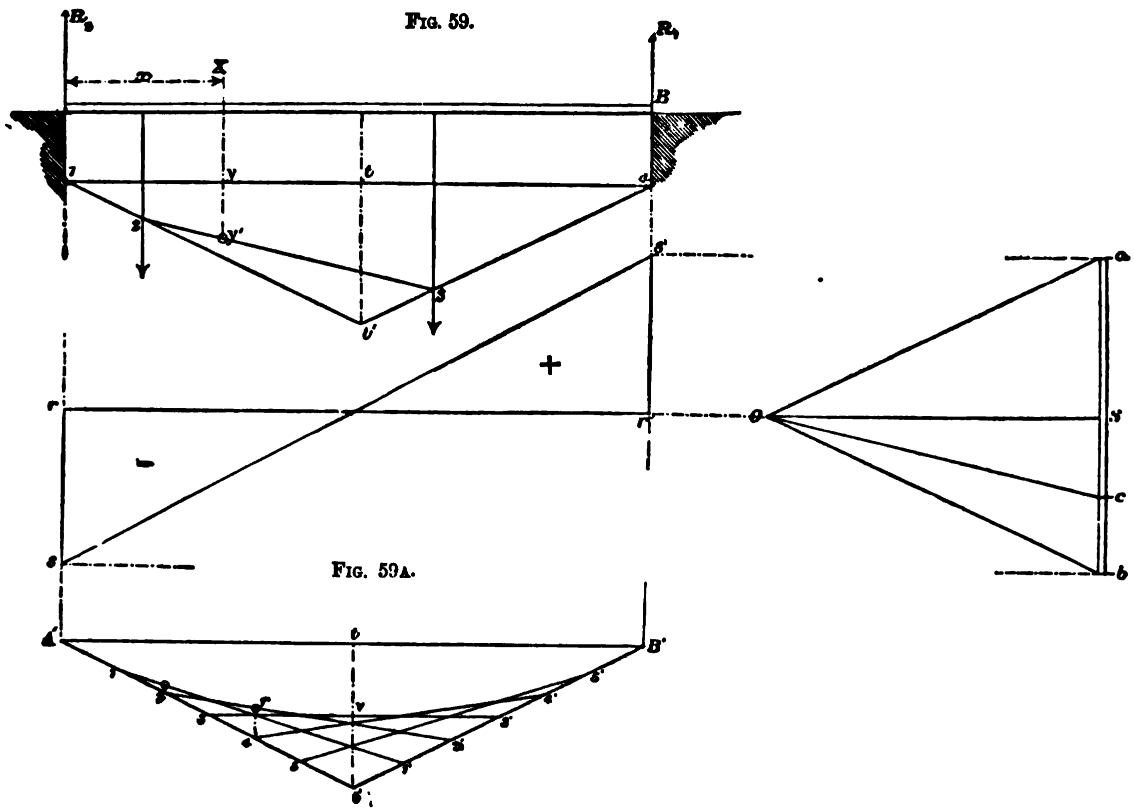


FIG. 59A.

fact be regarded as an infinite number of equal concentrated loads infinitely near together, and the parabola is the curve which the funicular polygon of such a system of loads becomes. Putting $x = o$, $x = l$, and $x = \frac{l}{2}$, it is evident that the bending moment at either abutment is o, and at the centre of the beam $- \frac{w \,.\, l^2}{8}$. The parabola in question is of the form A$'$ v B$'$ (Fig. 59A); the vertex is under the centre of the beam, and the axis is at right angles to the beam.

Draw the load line $a\,b$ representing $w \,.\, l$ (the total load), bisect $a\,b$ in S, and take any pole O on a line from S at right angles to $a\,b$. Draw A$'$ t' parallel to O b and t' B$'$ parallel to O a, then A$'$ t' B$'$ is the funicular polygon of the total load, and if $w \,.\, l$ acts as a concentrated load at the centre of the beam, the bending moment at the centre of the beam is (§ 44) $t\,t'$. H′

The triangle A′ t' t, O b S are similar.

Hence

$$\frac{t\,t'}{A\,t} = \frac{b\,S}{S\,o};\ \text{or,}\ t\,t' = \frac{l}{2}\cdot\frac{\frac{w\,l}{2}}{H} = \frac{w\,.\,l^2}{4\,.\,H}$$

The bending moment at the centre of the beam, supposing $w\,.\,l$ to act there as a concentrated load, and H to be taken as unity, is therefore $\frac{w\,.\,l^2}{4}$, or half the bending moment at the centre, when $w\,.\,l$ is uniformly distributed. Hence v, the vertex of the parabola, bisects $t\,t'$.

Any number of tangents to the parabolic funicular curve can readily be drawn: thus, to obtain a tangent at a point vertically below the section X (Fig. 59), set off $b\,c$ on the load line equal to $w.x$. (This will amount to dividing $a\,b$ in the same proportion as B A is divided at X, since the load distribution is uniform.)

The bending moment at X will not be altered by considering two concentrated loads $w.x$ and $w\,(l-x)$ to act at the middle points of A X and X B respectively. Join c O. Thus $y\,y'$, the ordinate below X of the funicular polygon 1 2 3 4 of these concentrated loads, gives (§ 45) the bending moment due to them at X: $y\,y'$ is also an ordinate of the parabolic funicular curve at y', and 2 3 is a tangent to this curve at y'. Any number of tangents and their points of contact can similarly be obtained; and following out the above principles, the following simple construction is arrived at.

Bisect A′ B′ (Fig. 59A) equal and parallel to the beam by a perpendicular, and draw A′ t', B′ t' parallel to the rays O b, O a. Divide A′ t' and B′ t' into any number of equal parts, 1 2 3 . . . ., 1′ 2′ 3′ . . . Join 1 1′ 2 2′, &c., as shown in the figure; the lines obtained are tangents to the parabola; and moreover, the point of contact of any tangent bisects the distance between the points in which adjacent tangents cut that tangent. Thus r, obtained by drawing a vertical line up through 4, is the point of contact of 2 2′.

A parabola of which the vertex, axis, and one point are known, can also be drawn as follows. From V, the vertex (Fig. 60), draw a line V T perpendicular to the axis V F. From B, the given point on the curve, draw B T parallel to the axis. Divide V T and B T into the same number of equal parts. Join each of the divisions of B T with the vertex, and cut the lines obtained by lines drawn parallel to the axis through the divisions of V T, as shown in the figure. To determine the focus, draw B X perpendicular to the axis; make V t = V X. Join B t, then B t is the tangent at B, and a line bisecting B t at right angles determines the focus F.

The following is a very convenient way of drawing the parabola approximately by arcs of circles if the vertex and focus are known. F_2 and V (Fig. 61) are the focus and vertex respectively of the required parabola. Draw any lines 1 2 3 . . . at right angles to the axis. It will be more convenient to divide V F_2 into equal parts, and draw the lines 1 2 3 . . . through the parts of division, continuing these lines at the same intervals. Make o_1V = 2 . V F_2, and with o_1 as centre and o_1V as radius, describe a circular arc as far as *a*, a point approximately on a line bisecting V 1 and perpendicular to the axis. With 2 as centre, describe an arc with radius 2 . V F_2 cutting the axis in o_2; with o_2 as centre and radius to *a* describe the arc *a b* as far as *b*, about half-way between the lines 1 and 2. With 3 as centre, describe an arc of radius 2 . V F_2 cutting the axis in o_3; with o_3 as centre and radius to *b*, describe the arc *b c*, and continue the process till the curve has been drawn as far as required. The symmetrical

FIG. 60.

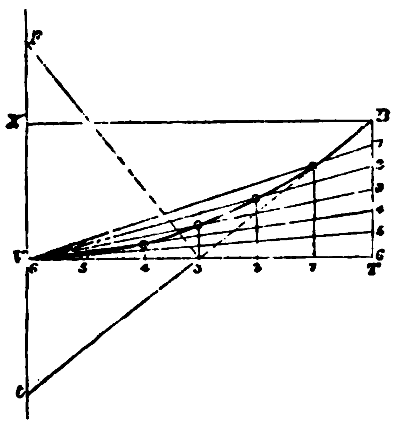
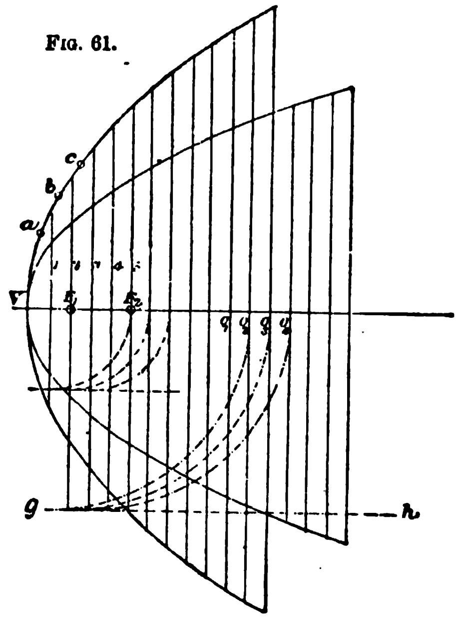


FIG. 61.



portion of the parabola on the other side of the axis is, of course, drawn at the same time. By drawing a line *g h* at starting parallel to and at a distance from the axis equal to 2 . V F_2, the construction is shortened, as the constant radius 2 . V F_2 employed in obtaining the centres o_2, o_3, &c., is then always ready to hand. If (as in the figure) the distance V F_2 is in the first instance divided into equal parts, and all the other ordinates drawn at equal intervals, the centres o_1 o_2 o_3, &c., require no further determination. If the lines 1 2 3 . . are not taken too far apart, the curve will approximate very closely indeed to the parabola. In the figure a second parabola with the same vertex V and F_1 as focus has been similarly drawn.*

In practice the parabola may be represented by a circular arc if *t v*, the ordinate of the vertex (Fig. 59A), is not greater than ⅛th of the span; but it is frequently necessary

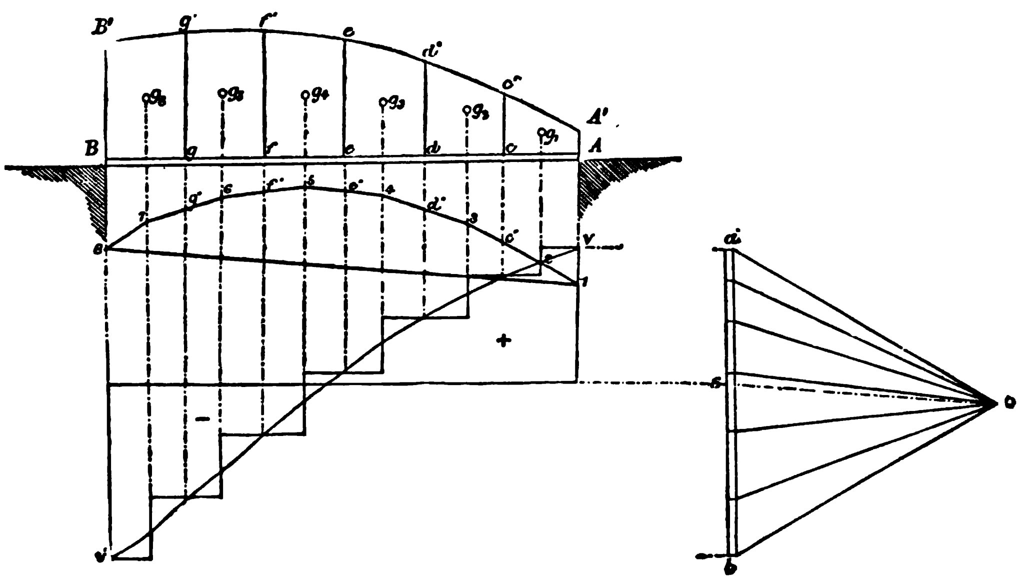
* This construction is due to Pröle.

to draw parabolas of greater eccentricity, in which case one of the two constructions given above should be employed. The parabola having been drawn, the ordinate under any section gives the bending moment at that section.

The case of a uniformly distributed load partially covering a beam will be treated under the head of travelling loads.

47. **Load with any Fixed Distribution.**—If the load has any distribution other than uniform, it will be necessary to represent it by an area (termed the *loading area*) bounded by a curve, the ordinate at any point of which represents the load per unit of length at that particular point. This area is then cut up into vertical strips, and the load represented by each strip is dealt with as if it were replaced by an equal concentrated load acting through the centre of gravity of the strip. Thus if the ordinates A A′, *c c′*, *d d′*, &c. (Fig. 62), represent the respective loads per unit of length at the

FIG. 62.



equidistant sections A, *c*, *d* . . . of the beam A B, the whole area A B B′ A′ will represent the load distribution. The strips are approximately trapeziums whose centres of gravity $g_1, g_2 \ldots$ may be taken to lie in their centre lines, and whose weights $w_1, w_2 \ldots$ are approximately proportional to their mean heights.

Draw *a b* representing the total load on the beam, and divide *a b* in the proportion of the mean heights of the six trapeziums. Join the points so obtained to any pole O, and draw the funicular polygon 1 2 . . . . . 8. The sides of this polygon are tangents to the funicular curve at the points 1, c'', d'' . . . g'', 8, the funicular curve can therefore be drawn, and its ordinates give the bending moment at all sections.

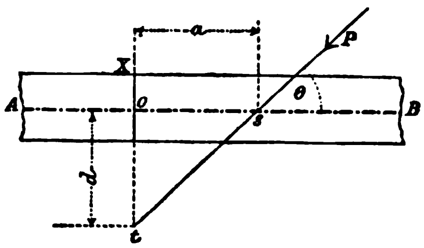
The shearing stresses, if the beam is considered as loaded by the concentrated loads $w_1, w_2 \dots$ acting through $g_1, g_2, \dots$ will be given (§ 45) by the ordinates of a figure made up of rectangles, and these stresses will for the sections $c, d, e \dots$ be evidently unaffected by the hypothesis of concentrated loads in place of continuous loading. Hence the ordinates of a curve $v\,v'$ drawn through points on the sides of the respective rectangles vertically under $c, d, e \dots$ (i.e. the middle points of these sides) will give the shearing stress at any section.

As in the previous figures, the horizontal through S separates positive and negative shearing stress, and shows when the change of sign takes place.

By cutting the loading area up into a larger number of strips a closer approximation to the actual conditions of loading is of course obtained. The curve A′ B is termed the *extrados* of the load.

48. **Forces in one Plane but not Parallel.**—Consider first the effect on a section X (Fig. 63) of a single force P acting in the plane of symmetry of a beam A B, and at an angle θ to the axis. The line of action of P cuts the axis in s, and the plane of the section in t. Suppose P to act at s, and to be replaced by its vertical and horizontal components. The former P sin. θ gives rise (§ 41) to a couple whose moment is a . P sin. θ, and to an unbalanced force P sin. θ acting in the plane of the section, and causing an equal and opposite shearing stress on that section. The horizontal component P cos. θ acts as a direct stress, or pressure on the section. It is the same if P is supposed to act at t; the horizontal component gives rise to a couple P whose moment is d . P cos. θ, and an unbalanced force P cos. θ causing direct stress on the section, while the vertical component P sin. θ causes an equal and opposite shearing stress on the section X. Since $d = a \tan. \theta$; $a\,\text{P} \sin. \theta = d\,.\,\text{P} \cos. \theta$.

Fig. 63.

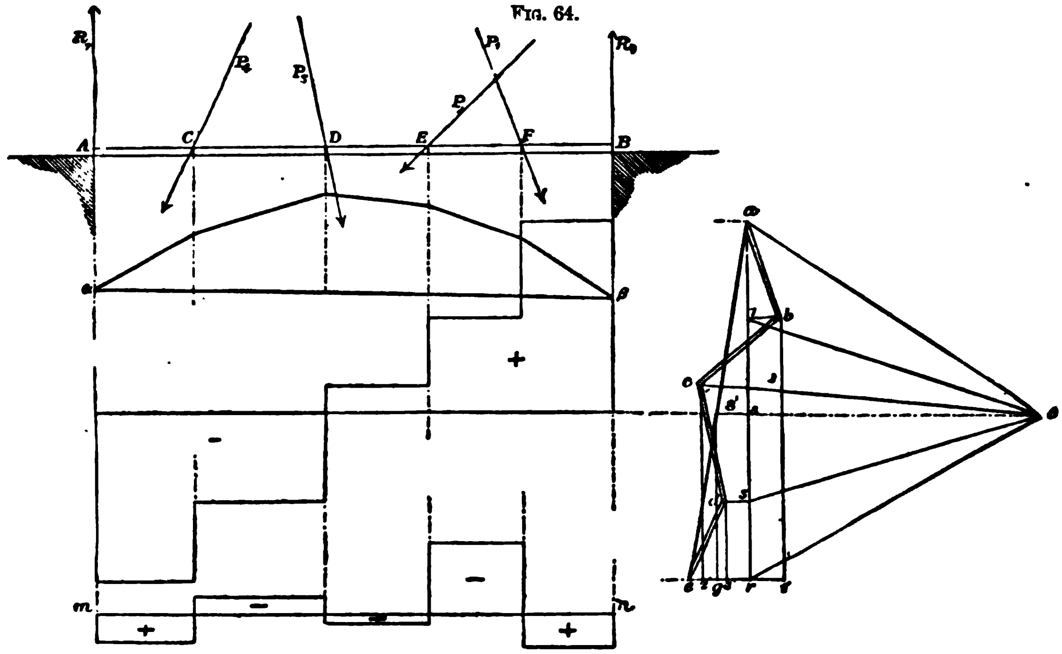


We may, therefore, deal only with the vertical components of the forces $p_1 \dots p_4$ (Fig. 64), and determine the bending moments due to these components acting as vertical loads at C .. F, leaving the direct stresses in the several segments A C, C D .... to be separately obtained.

Draw a vertical $a\,r$ through a, and horizontals through b, c, d, e, cutting it in 1 2 3 r. Thus a 1, 1 2, 2 3, 3 r are the vertical components of the forces, and $a\,r$ is that of their resultant. Draw a funicular polygon of the forces a 1, 1 2, 2 3, 3 r, with respect to any pole O; the shearing stresses and bending moments are then obtained as in § 45.

Draw the funicular polygon of the vertical components, then O s parallel to $a\,\beta$ determines the magnitudes $a\,s$, $s\,r$ of the vertical components of R_1, R_2 respectively. Drop verticals b 1′, c 2′, $d\,r$ from b, c, d, on $e\,r$, then r 1′, 1′ 2′, 2′ 3′, 3′ e are the horizontal components of the four forces p_1, p_2, p_3, p_4, and these components together with the components of the reactions form a system in equilibrium. The direct stress on any section is equal to the resultant of all the horizontal components acting on either side of that section, and is a pressure, or a tension, according as this resultant acts towards, or away from the section. Thus the stress in the segment A C is a pressure, and equal to $e g$ (the horizontal component of the reaction at A); in C D

Fig. 64.

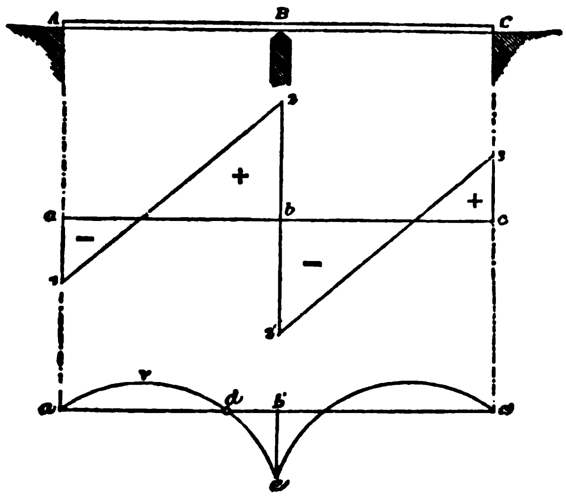


it is a tension, and equal to $e\,3' - e g = g\,3'$; in D E it is a pressure, and equal to $2'\,3' - g\,3' = 2' g$; in E F a tension, and equal to $2'\,1' - 2' g = g\,1'$; and finally in F B it is a pressure, and equal to $g\,1' - r\,1' = g r = s s'$, the horizontal component of the reaction at B. The distribution of direct stress in the beam can therefore be represented by a figure made up of rectangles as shown, the *depth* of the figure under any section representing the stress on that section. Pressures and tensions are shown on opposite sides of the line $m n$: the sum of the former is of course equal to the sum of the latter.

49. Beam resting on three Supports forming two equal Spans, and Uniformly Loaded.—*Shearing Stress.*—The beam A B C of length $2\,l$ (Fig. 65) rests on three supports at the same level, and sustains a uniformly distributed load w per foot run. By means of the "Theorem of three moments" it can be shown that the reactions of the supports are $\frac{3}{8} w . l$; $\frac{5}{4} w . l$; $\frac{3}{8} w . l$; hence making $a\,1 = c\,3 = \frac{3}{8} w . l$ and $b\,2 = b\,2' = \frac{5}{8} w . l$, and joining 1 2, 2′ 3, the diagram of shearing stress is obtained.

Bending Moment.—For any section distant x from A the bending moment is $\frac{3}{8}w \cdot l - w \cdot x\frac{x}{2} = w \cdot x\left(\frac{3\,l}{8} - \frac{x}{2}\right)$. Putting $x = o$, or $x = \frac{3}{4}l$, the bending moment vanishes; putting $x = l$, the bending moment becomes $-\frac{w \cdot l^2}{8}$. The curve of bending

FIG. 65.



moments for the span A B is a parabola passing through a', d and e, where $a'd = \frac{3}{4}l$ and $b'e = \frac{w \cdot l^2}{8}$, and the vertex of this parabola is at a horizontal distance $\frac{3}{8}l$ from a'. For the span B C the curve is of course an equal symmetrically situated parabola passing through e and c'.

CHAPTER VI.

SIMPLE BEAM.—TRAVELLING LOAD.

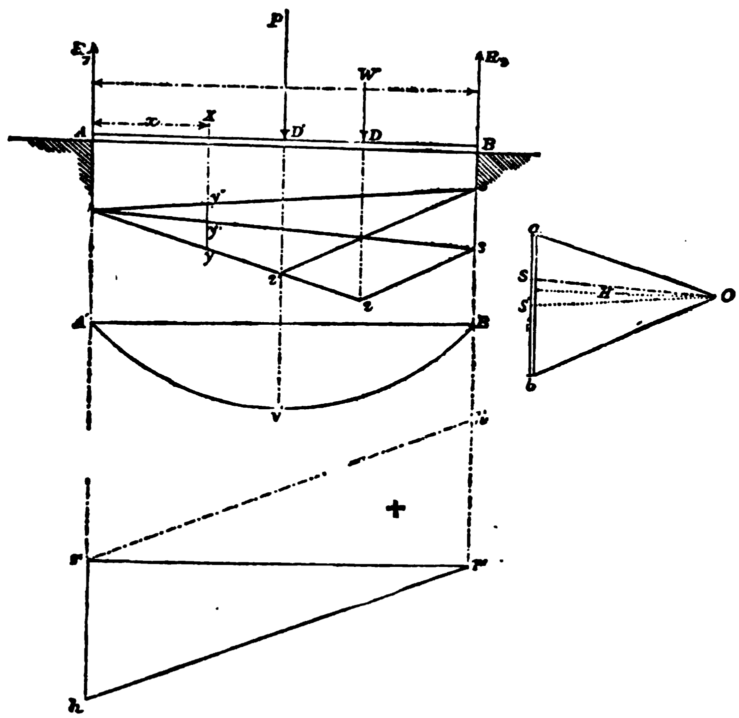
50. **Single Concentrated Load.**—*Shearing Stress.*—A beam must be sufficiently strong to resist the maximum shearing stresses and bending moments which can arise under any possible disposition of the given load. Hence, in dealing with a travelling load it is the maximum stresses and moments which have to be ascertained.

The beam A B (Fig. 66) sustains a load W acting at a point D to the right of a section X. The shearing stress at X is therefore (§ 44) equal to $-\text{W} \cdot \frac{\text{D B}}{l}$. Hence as

D B increases, or as W moves towards X, the shearing stress increases and is *negative*. If D is between A and X, the shearing stress at X is equal to $+W.\frac{AD}{l}$. Hence as W approaches X from the other abutment A, the shearing stress increases and is *positive*. The shearing stress at X is therefore a maximum when W is at X, and is positive or negative according as W approaches X from the right or left abutment.

Hence, if W moves from B to A, the successive maximum shearing stresses at the several sections of the beam are represented by the ordinates of a triangle $r r' h$, in which $r' h$ is equal to W, the shearing stress in this case being all negative. If W

Fig. 66.



moves from A to B, the maximum shearing stresses are all positive, and are represented by the ordinates of an equal triangle $r r' h'$.

Bending Moment.—Draw the funicular polygon 1 2 3 (Fig. 66) of the load W with respect to any pole O. Thus (§ 44) the bending moment at X, when W is at D, is $y' y.$ H. If W moves to the position D′, the funicular polygon takes the form 1 2′ 3′, and the bending moment at *X* becomes $y'' y.$ H. Hence evidently the bending moment at X increases as W moves towards X, and is a maximum when W arrives at X.

The maximum bending moment (M) at X is equal to $R_1 . x$, and

$$R_1 = w.\frac{l-x}{l}.$$

Hence $M = \frac{W}{l} \cdot x(l - x)$. The curve whose ordinates would represent the maximum moments at all sections of the beam is therefore a parabola $A' v B'$, and putting $x = \frac{l}{2}$, it is evident that the vertex v of this parabola is distant $\frac{W \cdot l}{4} \div H$ (or $\frac{W \cdot l}{4}$ if H is taken as unity) from $A' B'$.

51. **Any Number of Concentrated Loads.** — *Shearing Stress.* — If any load additional to W (Fig. 66) is applied between B and X, the shearing stress at X is evidently increased. If on the other hand the new load (P) is applied at a point D′ between A and X, the shearing stress is diminished, provided that P . A D′ is not greater than 2 . W . D B. If P . A D′ = W . A B, the shearing stress at any section between D′ and D is zero. As the product P . A D′ increases, the shearing stress at X increases, but changes sign. If P . A D′ = 2 W . D B, the shearing stress at X has the same value $\left(W \cdot \frac{DB}{l}\right)$ as before the addition of the new load P, but its sign has changed.

Hence, generally, if two or more equal or nearly equal loads preserving constant intervals move over a beam A B, the maximum shearing stress at any section X will arise when the leading load reaches X moving from the support farthest from X. In the case of a railway bridge, therefore, the maximum shearing stress at any section will usually occur when the longer segment is covered by the train, and the leading axle has reached that section.

A beam A B (Fig. 67, Pl. V.) of 45′ 0″ span is traversed by a train of passenger engines and tenders. The loads on the three engine-axles are 9, 15, and 7 tons; those on the three tender-axles are each 7 tons. The axle intervals are indicated. Supposing the train to move from right to left, the shearing stress at any section will be a maximum when either the leading or driving axle of the leading engine reaches the section. From a set off the loads $W_1, W_2 \ldots$ successively along the load line, making $a b$ equal to W_1, $b c$ equal to W_2, and so on. Draw the funicular polygon of the loads with respect to the pole O (Fig. 67a, Pl. VI.). (The polar distance H is made equal to 45 tons taken from the scale of loads.) It will be necessary for the funicular polygon to embrace the loads $W_1 \ldots\ldots W_8$, as some of the maxima shearing stresses (those for a length of 8′ 0″ from A) will evidently arise *after* W_1 has moved off the beam. To obtain the maximum shearing stress when W_1 arrives at a section X distant 15′ 0″ from A, take any point r on the line of action of W_1, draw $r l$ horizontal and equal to 15′ 0″, produce $r l$ to k, making $l k$ equal to A B. Drop verticals from l and k, cutting the funicular polygon in v and t respectively. Join $v t$, and draw the vector O s (Fig. 67a) parallel to $t\ v$, then (§ 44) the shearing stress at the section X is $a\ s$ when W_1 arrives at that section. To obtain the shearing stress when W_2 reaches X, take r' on the line of action of W_2, make $r'\ l'$ equal to 15′ 0″, and $l'\ k'$ equal to 45′ 0″. Drop verticals from l' and k' cutting the funicular polygon in v' and t', and draw the vector O s' (Fig. 67a) parallel to $t'\ v'$. Then $b\ s'$ (Fig. 67a) represents (§ 44) the shearing stress when W_2 arrives at X. This shearing stress is less than that arising when W_1 reaches X. Proceeding in this way, the maximum shearing stress at any section can be obtained. In Fig 67b, Pl. VI., the maxima shearing stresses for sections 1′ 6″ apart have been plotted, and the curve of maximum shearing stress has been drawn. This curve has cusps, which occur when each new load comes on the bridge at B, and also when the leading load W_1 goes off the bridge at A. The position of these cusps is indicated in Fig. 67b by dotted ordinates. The maxima shearing stresses will usually occur when W_1 arrives at any section; but for a portion of the beam A′ C (Fig. 67b) equal to the interval between W_1 and W_2 (8′ 0″) measured from A, the maxima stresses will be obtained by treating W_2 as the leading load. For some small distance to the right of C, W_2 arriving at a section will also be found to produce the maximum shearing stress.

For every negative maximum shearing stress there will of course be a corresponding positive maximum shearing stress at a symmetrically situated section when the same train traverses the beam from left to right.

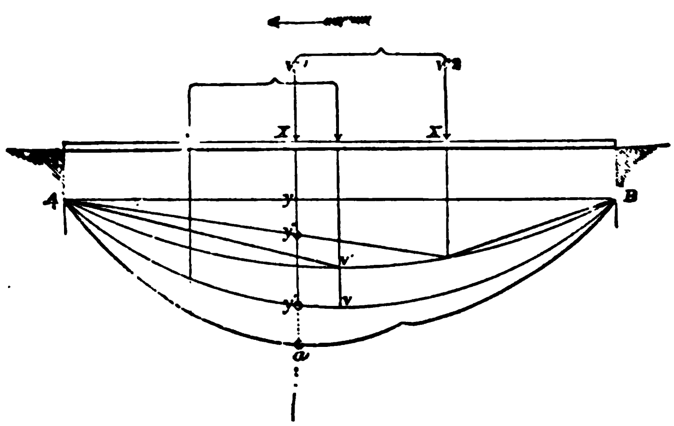
Bending Moments.—Referring to Fig. 66, it will be evident that any additional load applied *on either side* of the section X will increase the ordinate of the funicular polygon under that section, and therefore the bending moment. Moreover the maximum ordinate of any funicular polygon must evidently be at one of its angles. Hence, generally, the maximum moment at any section will occur when the heaviest load reaches that section, and the beam on both sides is as fully loaded as possible. It has been shown (§ 50) that when a single concentrated load traverses a beam, the curve of maximum moments is a parabola. If two or more loads preserving a fixed interval traverse the beam, the curve becomes much more complex; its ordinates may, however, be obtained by adding the ordinates representing the maximum moments due to the separate loads.* Thus, if A v B (Fig. 68) is the curve of maximum moments due to W_1, and A v' B that due to W_2, then when W_1 arrives at X, $y\,y'$ will (§ 50) represent the bending moment at X due to W_1; but when W_1 is at X, W_2 will be at X′, and the bending moment at X due to W_2 at X′ will be $y\,y''$. Hence the total bending moment at X due to W_1 at X and W_2 at X′ will be represented by $y\,a = y\,y' + y\,y''$. In this way ordinates representing the maximum moment at any section of the beam can be obtained, and a curve of maximum bending moment can be drawn.

* 'Wrought Iron Bridges and Roofs.'—UNWIN.

This curve can also be obtained by a method analogous to that above described for obtaining the curve of maximum shearing stress. Referring to Fig. 67, Pl. V., the maximum bending moments at any section X will occur when either W_1 or W_2 is at X, and the beam on both sides of X is as fully loaded as possible. In order, therefore, to obtain all possible conditions of loading, it will be necessary to extend the train by taking in a new load W_{10} corresponding to the trailing axle of the tender of an engine in front of W_1. The funicular polygon as altered to embrace this new load will take the form 1′ 2′ 2 3 4 . . . . Similarly new loads W_9 and W_{11} may have to be introduced at the other end of the beam, and the funicular polygon correspondingly extended.

To ascertain the maximum bending moment at a section X distant 15′ 0″ from A, draw *r l* equal to A X and *l k* equal to A B. Let fall verticals from *l* and *k* cutting the funicular polygon in *m* and *t*. Join *m t*; then *y* 2, the intercepted ordinate under *r*, represents the bending moment at X when W_1 is at X. To ascertain whether the bending moment at X is greater when W_2 is at X, draw *r′* 1′, *l′ k′* as before. Let fall verticals *l′ v′*, *k′ t′* from *l′* and *k′*, and join *v′ t′*. Then *y′* 3 is the bending moment at X when W_2 is at X; and since *y′* 3 is greater than *y* 2, the maximum moment at X arises when W_2 is at X. By precisely similar construction *y″* 8 is the bending moment when W_7 is at X′, and *y‴* 9 when W_8 is at X′. Hence *y‴* 9, which is greater than *y″* 8, is the maximum bending moment at X′. In this last construction W_7 and W_8 have been used in place of W_1 and W_2, as otherwise the condition of maximum loading on each side of X′ would not have been satisfied.

FIG. 68.

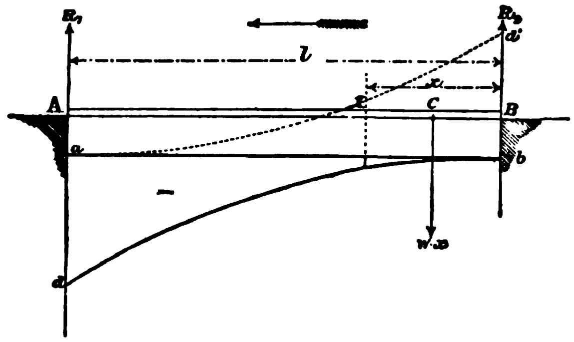


The maxima moments for sections 1′ 6″ apart have been thus obtained and plotted in Fig. 67c. The resulting curve is the curve of maximum bending moment when the given train traverses the bridge from right to left. This curve is not necessarily symmetrical; hence, for trains running both ways, the maximum bending moment will be represented by the *longer* of the ordinates under any two symmetrically situated sections.

It should be noticed that the principle involved in the above construction is the moving of the beam under the loads. The funicular polygon having been once drawn for all the loads, this movement of the beam gives a new position of its closing line, and on the position of the closing line of the funicular polygons, both shearing stresses and bending moments depend.

52. **Uniformly Distributed Load.**—*Shearing Stress.*—The beam A B (Fig. 69) of length l supports a uniformly distributed travelling load of w per foot run. If the load comes on the beam at B, the shearing stress at a section X distant x from A will be a maximum when the length X B is covered by the load, and will not be altered if a concentrated load equal to $w.x$ acting at C, the middle point of X B, is supposed to replace the uniformly distributed load.

FIG. 69.



The shearing stress at X is therefore

$$w.x - R_2 = R_1$$

and

$$\frac{R_2}{R_1} = \frac{AC}{CB} = \frac{2l - x}{x}$$

Hence

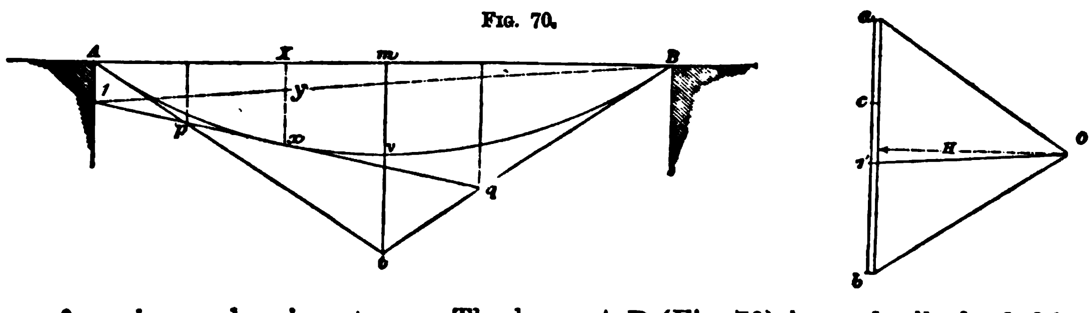
$$R_1 = \frac{w}{2l} \cdot x^2.$$

The curve of maximum shearing stress is therefore a parabola. Putting $x = o$ and $x = l$ the maximum shearing stress at B and A are o and $\frac{w \,.\, l}{2}$ respectively. If the scale of loads is such that $\frac{w \,.\, l}{2}$ is represented by a length $a\,d$ not greater than $\frac{l}{4}$, the circular arc through b and d having its centre vertically below b will approximate sufficiently closely to the parabola. If otherwise one of the methods given in § 46 may be employed. The shearing stress when the load moves from right to left is all negative. If the load moves from left to right, the shearing stress is all positive and may be represented by an equal parabola $a\,d'$, having its vertex at a and axis on $d\,a$ produced.

Bending Moments.—The maximum bending moment at any section will occur when the whole beam is fully loaded. The curve of maximum bending moment is therefore a parabola (§ 46).

Shearing Stress, obtained from curve of Bending Moments.—It is sometimes convenient to obtain the maximum shearing stress at any section without drawing the

FIG. 70.



curve of maximum shearing stress. The beam A B (Fig. 70) is gradually loaded by a travelling load of w per foot run, which moves from B to A. Draw the load line

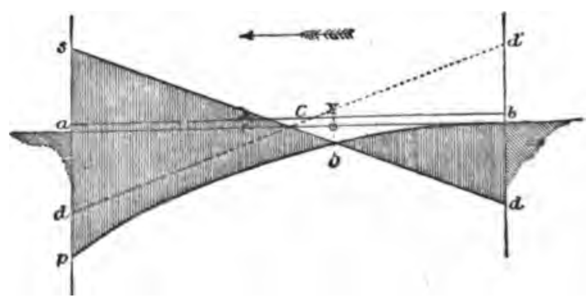
$a\,b$ and bisect $a\,b$ by a perpendicular H, making H equal to any convenient whole number taken from the scale of loads. Draw A t, B t parallel to a O, O b, then A t, B t are tangents to the funicular curve of the *whole* load at A and B, and v the middle point of $m\,t$ is its vertex (§ 46). The maximum shearing stress at a section X will occur when the load covers the longer segment B X. Bisect A X, X B by verticals cutting A t and B t in p and q. Then $p\,q$ is a tangent to the funicular curve of the whole load at x vertically under X. In order to cut away the portion of the total load covering the segment A X, produce $q\,p$ to cut the vertical through A in 1; join 1 B and draw 1′ O parallel to 1 B. Set off from a along the load line, $a\,c$ equal to the load on the segment A X $\left(\frac{a\,c}{c\,b} = \frac{\text{A X}}{\text{X B}}\right)$. Then c 1′ is the maximum shearing stress at X. In the same way the maximum shearing stress at any other section can be obtained. The bending moment at X when the load reaches that section is $y\,x$. H.

53. **Combined Stationary and Travelling Load.**—A railway bridge girder is subject to stress due to its own weight together with that of the platform and rails which constitutes its stationary or permanent load, and is sometimes assumed to be uniformly distributed. It is further subjected to stress due to passing trains which constitute its travelling load, and this load is almost always assumed to be uniformly distributed, an hypothesis which in bridges of considerable span introduces very slight error.

Shearing Stress.—By combining the diagrams of shearing stress for a stationary and travelling load, the actual *total* shearing stress is obtained.

Thus, in the case of a uniformly distributed load moving from right to left over a girder which also sustains a uniformly distributed dead load, the diagrams of shearing stress are (§ 52 and § 46) a parabola $b\,p$ and two triangles $a\,s\,c$ and $b\,d'\,c$ (Fig. 71). The triangle $a\,s\,c$ represents negative shearing stress, thus in order that the ordinates of the triangles and parabola may be conveniently summed, the position of the negative triangle should be reversed with respect to $a\,b$. The ordinates of the hatched figure $s\,p\,o\,b\,d$ then give the total shearing stress. The shearing stress changes sign at X and is negative from a to X, and positive from X to b. If the figure is drawn for a uniformly distributed load moving from left to right, the change of sign will take place at a section X′ symmetrically situated with respect to the centre of the girder. Hence in a railway girder there is a portion X X′ at the centre in which the shearing

FIG. 71



stress is positive or negative according as the load comes on the bridge from the left or right abutment. It is this peculiarity in the distribution of shearing stress which necessitates counter-bracing near the centre of a girder, inasmuch as the bracing bars may here be called upon to act as struts and ties alternately.

Similarly, by combining the shearing stress diagrams for a system of concentrated stationary loads (§ 45) with that for a system of concentrated travelling loads (§ 51), or with that for a uniformly distributed travelling load (§ 52), the maximum total shearing stress at any section due to these respective conditions of loading can be obtained.

Bending Moment.—The maximum total bending moment at any section will be simply the sum of the ordinates of the funicular polygons or curves of the stationary and travelling loads, provided of course that these polygons or curves have been drawn with the same assumed polar distance. By summing these ordinates a curve of maximum total bending moment can be drawn if required.

54. **Curve of Total Stress in Booms.**—It is sometimes desirable in dealing with girders to make use of a curve the ordinates of which represent the total stress at any section of either boom. This curve of total stress will, for a girder of uniform depth, evidently be of the same form as that of total bending moment, since the total stress in the booms at any section is equal to the total bending moment at that section divided by the depth of the girder. The one curve can therefore be used in place of the other by a mere adjustment of scales. If the depth varies—as in the case, for example, of a "hog-backed" girder—the curve of total stress must be deduced from the curve of bending moment.

CHAPTER VII.

BRACED GIRDERS.

55. **General Considerations.**—In a perfect braced structure each member is subject only to direct tension or pressure, so that the total stress is the same for any cross section of a given member, and its intensity is uniform over that section. In dealing with braced structures, therefore, it is usual to assume that there is a joint-pin at all the meeting points of the bars, and that the axis of this pin passes through the intersection of the axes of the bars. The travelling load and the weight of the platform are usually carried by cross girders, attached at the joints of one of the booms; the girder is therefore supposed to be loaded only at these joints. The weight of the girder itself is also supposed to act at the joints of the booms, and if this weight is considerable, it should be supposed to be equally divided among the joints of both

booms. In small girders the *whole* load may be supposed to be borne on the same set of joints. The proportion of load to be borne on each joint will therefore depend on the distance apart of the cross girders.

56. **Nature of the Loads on a Railway Girder.**—It is usual to reduce the concentrated loads to an equivalent uniformly distributed load. On short spans this average load will be greater than on large spans, and the stresses in the former case will be greater in consequence of the concentration of the actual loads at points whose distances apart are small, relatively to the span. A higher average should therefore be taken for small span girders. The following may be adopted, viz. 2 tons per foot run of each line of rail for 25′ 0″ span; $1\frac{3}{4}$ ton for 30′ 0″; $1\frac{1}{2}$ ton for 40′ 0″; and $1\frac{1}{4}$ for 60′ 0″ and upwards. The platform* is either carried on cross girders at short intervals which receive the load from timber sleepers, or the cross girders receive their load from longitudinal girders under each railway bar, whose length is a multiple of the length of the bay of the loaded boom of the main girders. If the former, the load on the cross girder should be taken to be that on the most heavily loaded axle of the heaviest locomotive (about 15 tons), multiplied by the number of lines of rail supported by a pair of main girders. This load will be concentrated at points whose distances apart will depend upon the gauge. If the latter, the load on the longitudinal girders may be taken as $2\frac{1}{4}$ tons per foot run of each line of rail, the loads on the cross girders will then be equal to the total load on the longitudinal girders, and will be concentrated at the points of attachment of the latter. The curves of shearing stress and bending moment for longitudinal girders may therefore be drawn by § 52. No general method can be given for the determination of the stress in girders subject to travelling loads, but it may be noticed that the stress diagram can usually be employed to determine the stresses in the *booms* which attain a maximum when the girder is fully loaded. The stress diagram can also be employed to determine the stresses due to the *dead load*, these stresses being afterwards added to the stresses due to the travelling loads. When only two bars meet at the end joints of the girder, the stresses in these bars can be obtained by a direct resolution along their directions of the reaction of the abutment when the girder is fully loaded.

57. **Warren Girder.**—*Load on one set of Joints.*—Fig. 72 represents the frame diagram of a Warren girder of 50′ 0″ span, made up of five 10′ 0″ bays. The travelling load is taken at $1\frac{1}{2}$ ton per foot run, the weight of the girder, platform, &c., at 1 ton per foot run. The whole of the load is supposed to be borne on the joints of the lower boom, and there is a cross girder at each joint. The bridge is supposed to consist of two girders carrying a double line of rail. Thus, supposing each joint to sustain a portion of the load extending as far as the centres of the adjacent bays,

* For weight of platform, &c., see Appendix.

there will be 10 tons of dead load and 15 tons of travelling load at each of the four bottom joints.

Stresses in Bracing Bars.—It can easily be shown that the stress in any bracing bar is a maximum when the shearing stress in the bay to which the bar belongs is a maximum, or (§ 51) when the bridge is fully loaded on one side of that bay. Since, in the present case, only the bottom joints are loaded, the shearing stress will be uniform over each of the several bays A 2, 2 4, &c., of the bottom boom. Thus, any pair of bracing bars 2 3, 3 4 will be in a condition of maximum stress when the joints 4, 6, and 8 are fully loaded.* Similarly the bars A 1, 1 2 will be in a condition of maximum stress when the joints 2, 4, 6, 8 are fully loaded. In thus considering each joint to be suddenly loaded as the train moves over the bridge, a supposition is made which, if not in accordance with the actual condition of things, is in favour of the bridge, since obviously as each point becomes loaded, the joint immediately in front of it will take up some of its load. Since the maximum shearing stress is uniform over each bay, it is evident that the distribution of maximum shearing stress during the passage of the travelling load from B to A will not be represented by a curve as in § 52, but a stepped polygon, the depth of which will vary for each bay. To draw this polygon make $a\,v$ equal to the travelling load on one joint (15 tons), join $v\,b$ and let fall verticals through all the joints 2 4 . . of the lower boom, the vertical through 8 cutting $a\,b$ in c and $v\,b$ in c'. Then $c\,c' = a\,v \,.\, \frac{c\,b}{a b}$ or $c\,c'$ represents the shearing stress from 8 to A due to 15 tons at 8. Draw a horizontal through c' to cut the vertical through 6 in d, the same vertical cutting $a\,b$ in e and $v\,b$ in e'. Now $e\,e'$ is the shearing stress from 6 to A due to 15 tons at 6, make $d\,d'$ equal to $e\,e'$; then $e\,d'$ represents the maximum shearing stress in the bay 6 4. Draw a horizontal through d' to cut the next vertical and proceed in the same way till the stepped outline is complete.

This outline can also be obtained from the parabolic curve of maximum shearing stress due to a uniformly distributed travelling load (§ 52). Make $a\,p$ equal to the half-total travelling load over the whole bridge $\left(\frac{75}{2} = 37\frac{1}{2} \text{ tons}\right)$, and by mean of the method of § 46 draw the parabola $b\,p$. If then verticals are dropped from the middle points of the bays of the lower boom to cut the parabola in $s_1\,s_2\,s_3$, then horizontals through $s_1\,s_2\,s_3$, &c., terminated by verticals through A 2 4, &c., give the stepped figure as before.

In order now to combine the shearing stress due to dead load with that due to

* While the load passes from 4 to 2 the stresses in the bars 2 3, 3 4 are *increased* by the loading up of 4 and *decreased* by the loading up of 2. These stresses, therefore, will never exceed those obtained by assuming 4 fully loaded and 2 without any load.

travelling load, make $a\,r$, $b\,r'$ equal to half the total dead load on the girder, and join $r\,r'$. Then (§ 46 and § 53) the ordinates of the figure contained between the line $r\,r'$ and the stepped outline $b\,c\,c'\,d\,d'$, &c., represent the maximum total shearing stresses at all sections of the girder: thus the ordinate $s_2\,t_2$ represents the maximum total shearing stress at a vertical section through 3. Draw $s_2\,h_2$, $s_2\,g_2$ respectively parallel to 32, 34, and

FIG. 72.

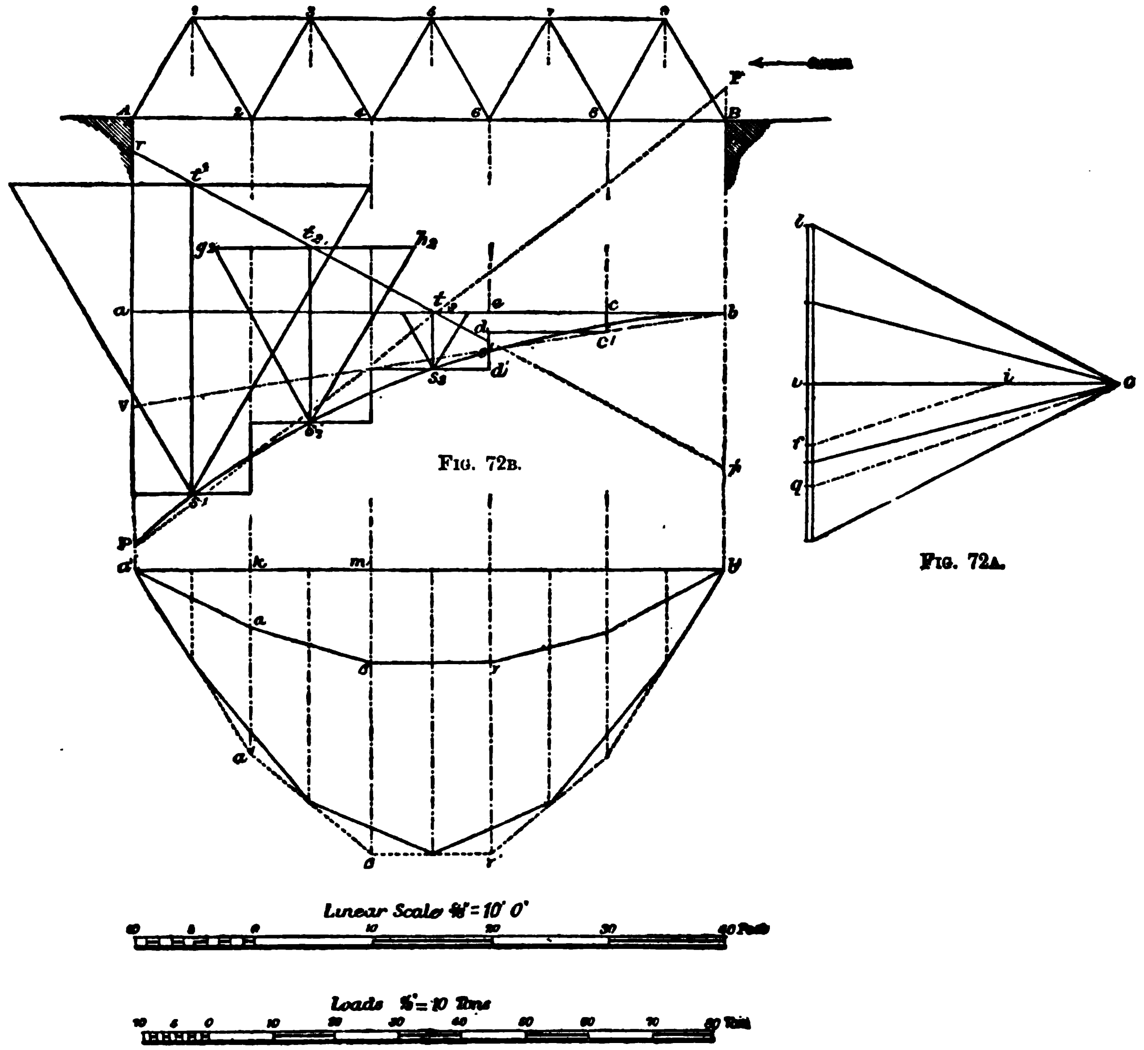


FIG. 72B.

FIG. 72A.

terminated by a horizontal through t_2, then $s_2\,h_2$, $s_2\,g_2$ represent the maximum stresses in the bars 23, 34; the former a pressure, the latter a tension. Similarly the stresses in all the bars can be obtained, but it will not be necessary to carry on the construction beyond 6, the point at which (§ 53) the shearing stress changes sign. Starting from the other abutment B, the stress determined for the bar A 1 must be written against the corresponding bar B 9 as its maximum stress when the travelling load comes on the girder at A, and similarly for the other corresponding bars of the right half girder. It may therefore happen that at the centre of the girder there are bars which have both a tension and a pressure written against them, showing that in these bars the nature of the maximum stress depends upon the direction in which the load moves. Such bars must be designed to resist both kinds of stress.

Stresses in Booms.—Since the stress in both booms will be a maximum when the whole girder is fully loaded, a stress diagram can be drawn by the methods of Chapter IV., the sides of which, in accordance with the principle of reciprocal figures, will give the maximum stresses in the several segments of the booms. Such a diagram is shown in Fig. 48A, p. 48, the only difference being that the upper joints only are supposed to be loaded, instead of the lower joints as in the present case. This method will in some cases be found to be the quickest, while in others it is more convenient to obtain the maximum stresses in the booms from the curve of maximum bending moment. In a girder loaded at the joints only, the distribution of maximum bending moment will be represented by a polygon instead of a curve, as in § 52. This polygon can be obtained from the curve of maximum moments due to a uniformly distributed load (§ 52), but it will be quicker to draw the funicular polygon at once. Thus, on a load line $l\,n$, (Fig. 72A) set off the loads* on half the girder, draw $n\,O$ at right angles to $l\,n$, and make $n\,O$ equal to any convenient whole number taken from the scale of loads; draw the funicular polygon $a'\,a\,\beta\,\gamma$. . (Fig. 72B) of the loads with respect to the pole O. Now the bending moment on a vertical section of the girder through the joint 4 is equal to $m\,\beta\,.\,n\,O$ (§ 45): taking moments about 4, it is evident that the stress in the segment 35 of the upper boom multiplied by h, the depth of the girder must be equal to the bending moment at the joint 4. Hence the stress in 35 is equal to $\frac{m\,\beta}{h}\,.\,n\,O$., and can be readily calculated or obtained by construction as follows:—

Make $n\,q$ equal to h, join $q\,O$, make $n\,f$ equal to $m\,\beta$, and draw $f\,i$ parallel to $q\,O$; then $n\,i = \frac{m\,\beta}{h}\,.\,n\,O$ = maximum stress in the segment 35 of the upper boom. Make $m\,\beta'$ equal to $n\,i$; then, since h and $n\,O$ are constant, it is only necessary to make $\frac{k\,a}{a\,a'} = \frac{m\,\beta}{\beta\,\beta'}$ in order to obtain $k\,a'$, the stress in the segment 13 of the upper boom. By thus multiplying all the ordinates of the angles of the funicular polygon by the constant $\frac{n\,O}{h}$, a new polygon $a\,a'\,\beta'\,\gamma'$ is obtained. This is the *polygon of total stress* (§ 54) for both booms, and the ordinate of this polygon under the middle point of any segment of either boom gives the stress in that segment, all the stresses in the upper boom being pressures and in the lower boom tensions. The above construction need only be carried out for half the girder, as symmetrically situated segments of the booms will

* The loads are plotted to *half* the given scale in order to keep down the size of the funicular polygon.

undergo similar stresses. It may be noticed that in the present cases the stresses in 35, 57, and 46 are equal.

The stresses in the booms may also be determined from the diagram of total shearing stress. Join p to t_3, the middle point of $a\,b$ (Fig. 72), and produce $p\,t_3$ to p'; then the ordinates of the figure made up of the two triangles $r\,p\,t_3$ and $r'\,p'\,t_3$ will represent the shearing stresses at all sections of the girder when the latter is completely loaded with 1 ton per foot dead load and 1½ ton per foot travelling load. Dealing with the ordinate of this new figure under the joint 3 precisely as before, fresh values will be obtained for the stresses in the bars 32 and 34. The horizontal components of these stresses will be given directly by the figure, just as $g_2\,t_2\,h_2$ are the horizontal components of the stresses in 34, 32 when only the joints 46 and 8 are loaded with the travelling load. Similarly the horizontal components of the new stresses in all the bracing bars of the left half-girder can be obtained. Now the stress in the segment of the boom A 2 is equal to the horizontal component of the stress in A 1, and the stress in 12 is equal to the sum of the horizontal components of the stresses in 12 and A 1, and generally the stress in any segment of either boom is equal to the sum of the horizontal components of the stresses in all the bracing bars between that segment and the nearest abutment.

The *total maxima* stresses in all the bars having been obtained and carefully tabulated, the dimensions of the bars must be calculated. The *weights* of the latter can now be ascertained, and the stresses should be redetermined, taking the weight of the girder itself into consideration.

58. **Warren Girders Loaded at all Joints.**—Taking the weight of the girder (Fig. 73) at 0·4 ton, the weight of the platform, platform girders, rails, &c., at 0·6 ton per foot, and supposing one-half the weight of the former to be borne on the upper joints 1 3 5 7 9, there will be—

	Tons.
Dead load on each upper joint	2
" " lower "	8

the travelling load remaining as before, 15 tons on each lower joint.

Stresses in Bracing Bars.—The effect of this transference of a portion of the load to the upper joints will be that the shearing stress will no longer be uniform over each of the bays A 2, 24, &c., but will now be uniform only under each half bay or under each bracing bar. Draw the stepped outline $s_1\,s_2\,s_3 \ldots b$ representing the distribution of maximum shearing stress due to live load. This outline will be, of course, precisely the same as that in Fig. 72. Make $a\,d$ (Fig. 73) equal to half the total dead load, and set down successively from d along $d\,a$ the loads at 1, 2, 3, 4. . . . Then by drawing horizontals through the divisions on $d\,a$ thus obtained, a new stepped outline $d\,e\,e'\,d'$ is arrived at, the ordinates of which give the shearing stress due to dead load (§ 45).

Thus the ordinates of the figure comprised between the two stepped outlines give the maximum total shearing stresses at all sections of the girder, and $d\,s_1$ is the maximum

Fig. 73.

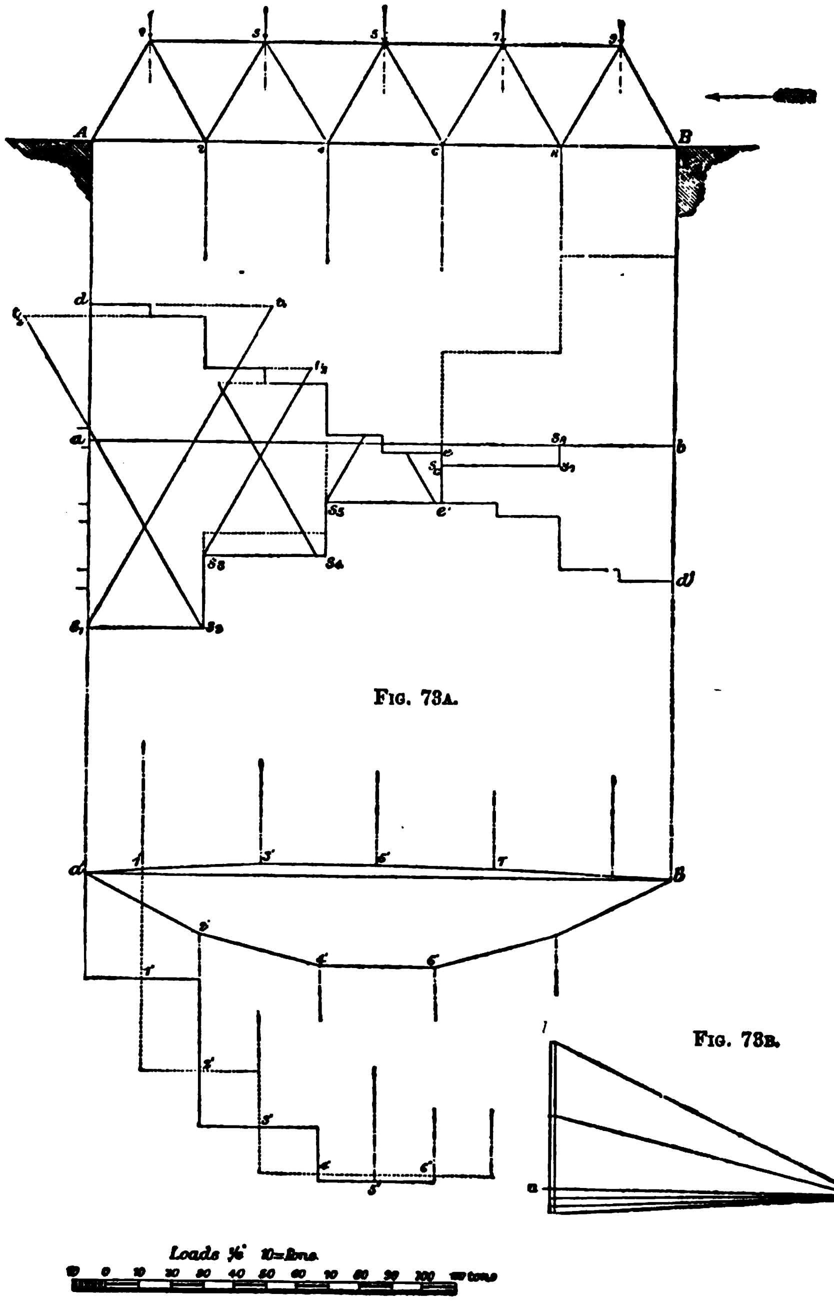


Fig. 73A.

Fig. 73B.

total shearing stress in the left half of the bay A 2 (Fig. 73). Draw $s_1\,t_1$ parallel to the bar A 1 terminated by the horizontal $d\,t_1$, then $s_1 t_1$ is the maximum stress in A 1. Similarly $s_1\,t_2$ and $s_3\,t_3$ are the stresses in the bars 12, 23 respectively.

Stress in Booms.—On the load line $l\,n$ (Fig. 73B) set off successively the loads on the lower joints of half the girder, and from n on $l\,n$ produced set off the loads on the upper joints of half the girder; thus the first load set off from n will be half the load at 5, or 1 ton. Take a pole O, such that n O is at right angles to $l\,n$, and equal to a convenient whole number taken from the scale of loads. Draw the two funicular polygons $a\,2'4'\ldots$ and $a\,1'\,3'\ldots$ as shown in Fig. 73A; the polygon $a'\,3'\,5'$ representing the distribution of bending moment on the upper and $a\,2'\,4'\,6'$ on the lower boom. The ordinates of the figure comprised between these two polygons will represent the bending moments over the whole girder. By multiplying the ordinates of this figure under each joint by $\frac{n\,O}{h}$ as before, and setting down from $a'\,b'$ (Fig. 73B) the lengths obtained, the points $1''$, $2''$, $3''\ldots$ are arrived at. By joining $a'\,2''$, $2''\,4''$, $4''\,6''$ the polygon of total stress of the lower boom would be obtained, and similarly $1''$, $3''$, $5''$ is the polygon of total stress for the upper boom. Then the maximum stress in any segment of either boom is given by the ordinate under the middle point of that segment of the polygon of total stress of the boom to which it belongs.

Substitute in Fig. 73 for the outline $s_1\,s_2\,s_3\ldots$, &c., a new outline (shown in dotted lines) representing the distribution of shearing stress when the whole of the bridge is covered by the travelling load; then new values for the stresses in the bracing bars will be obtained, and by summing the horizontal components of these stresses as described above the stress in any segment of either boom can be arrived at.

The stresses in the booms can also be obtained by an ordinary stress diagram.

If in a girder of this kind the travelling load is transferred from longitudinal girders to cross girders attached say at alternate joints of the lower boom, this will modify the outline of the figure representing the shearing stress due to travelling load, as the shearing stress due to travelling load will be uniform between the points of application of the latter, i.e. between alternate joints instead of adjacent joints, as in the case dealt with above.

59. **Lattice, or Trellis Girder.**—Fig. 74 represents a lattice girder of 50′ 0″ span. The bracing consists of four systems of triangles. The angle of inclination of the bars is 45°. The travelling load is taken at 1 ton per foot, dead load due to platform, rails, &c., 0·5 ton per foot, dead load of girder 0·4 ton per foot. The latter is supposed to be carried half on the lower and half on the upper joints. The booms are divided at the joints into 12 bays of 5′ 0″. The loads are therefore as follows:—

	Tons.
Dead load on upper joints	1
" lower "	3·5
Travelling load on lower joints .. .	5·0
Total " "	8·5

Stresses in Bracing Bars.—It will be necessary to deal with each system of triangulation separately. Selecting the system 1 2 3 4 5 6 7 8, set off the *dead* loads on the joints 2.....7 successively on a load line, and by drawing horizontal lines through the divisions on this line to meet verticals through the loaded joints, the shearing stress diagram due to dead load is obtained (§ 45). On account of the unsymmetrical distribution of the loads, the horizontal line separating positive and negative shearing stress will not bisect the load line, but must be found by drawing the funicular polygon of the dead loads with respect to any pole, as in § 47. Now draw the shearing stress diagram due to the travelling loads at 2 4 and 6, supposing them to come on the bridge from the right abutment. The construction is precisely similar to that of § 57. The resulting figure, combined with the previously drawn shearing stress diagram for dead load, represents the distribution of total shearing stress along the girder, when only the numbered joints are loaded, and the travelling load comes on the bridge from the right. Then lines drawn parallel to each bracing bar and terminated by the sides, or produced sides, of the diagram of total shearing stress under that bar, give the maximum stresses in those bars. These tresses can now be tabulated as far as the bar 45, after which the shearing stress changes sign, the corresponding bars beginning at the right abutment will have the same respective maxima stresses when the load comes on the bridge from the left.

The maxima stresses in the bracing bars forming the remaining systems of triangulation can similarly be obtained.

Stresses in Booms.—The maxima stresses in the booms arise when the girder is fully loaded. Draw therefore the diagram of shearing stress due to travelling load acting at all three of the lower joints of the system of triangulation under consideration (§ 57). From this diagram the stress in any bracing bar, when the travelling load acts at all the lower joints, is obtained, and it is only necessary to add the stress due to dead load only, taken from the other diagram. The horizontal component of this stress should be obtained and written down. The horizontal components of the stresses in all the bracing bars having been written down, the maximum stress in any segment of either boom will be equal to the sum of the horizontal components of the stresses in all the bracing bars between that segment and the nearest abutment.

If the bracing bars are inclined at 45°, as in the present case, the horizontal component of the stress in any bracing bar will be equal to the shearing stress in the bay under that bar.

The maxima stresses in the booms may be approximately obtained from a curve of total stress drawn for a uniformly distributed load covering the whole girder. This uniformly distributed load amounts to 1·9 tons per foot run, the total load on the girder is therefore $1 \cdot 9 \times 60 = 114$ tons, this curve of total stress is a parabola (§ 54).

Set off along a load line $a\,d$ a length representing the total load (114 tons), bisect $a\,d$ by a perpendicular S O making S O equal to h the height of the girder (Fig. 74). Draw $a\,t$, $b\,t$ parallel to a O, O d (§ 46), and bisect $t\,c$ in v. Draw a parabola with v as vertex and $t\,b$, $t\,a$ tangents at b and a respectively (§ 46): then the bending moment N at any section X is equal to $y \cdot$ S O, but the *total stress* in either boom at a vertical section through X is $\frac{M}{h}$ or since S O $= h$, the total stress in either boom at X is equal to y.

The ordinates of the parabola $a\,v\,b$ (Fig. 74) read off from the scale of loads therefore give the total stress in both booms at all sections along the girder, and the

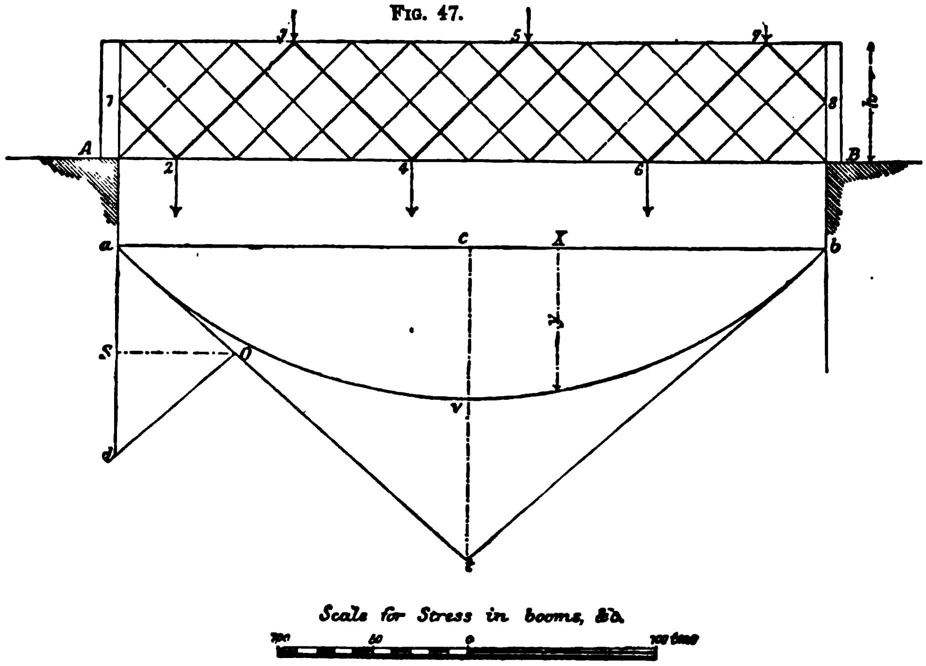


Fig. 47.

maximum total stress in any segment of either boom is given by the ordinates of the parabola vertically under the middle point of that segment. The stresses obtained will be very approximately those arising under the actual conditions of loading, and the approximation will be closer as the reticulations into which the web of the girder is cut up become smaller, or as the ratio between the length of a bay of the boom to the total span becomes larger.

The above is given merely as an example of graphic treatment, which is, nevertheless, in this case too tedious for actual practice. In the case of the lattice girder, arithmetical methods are simpler and more expeditious.

60. **Bowstring Suspension Girder.**—Fig. 75, Pl. VII., represents the frame diagram of one of the girders of Sarpsfos Bridge, Norway. The span is 60′ 9″, made

up of nine bays of 6′ 9″. The travelling load is taken at 1 ton per foot; weight of girder, 0·2 ton per foot; and of platform, 0·4 ton per foot. Supposing the upper joints to carry half the weight of the girder only, there will be—

	Tons.
Dead load on upper joints	0·675
" lower "	3·375
Travelling load on lower joints	6·750
Total " "	10·125

The diagonals are supposed capable of taking up tensile stress only. In the preceding paragraphs no allowance was made at starting for the weight of the girders themselves. In the present instance it is proposed to make such an allowance, but the remark at the end of § 57 will still apply. The stresses having been obtained, the *corrected weight* of the girder is arrived at, and the stresses should be redetermined for this corrected weight, unless it proves to be very near the original estimate.

Stress in Diagonals.—Make $a\,s$, $b\,s'$ (Fig. 75A), each equal to half the total dead load, join $s\,s'$. Draw below $a\,b$ the stepped outline (§ 57), which forms the shearing stress diagram for the travelling load coming on the bridge at B, and loading each joint successively. Then the ordinates of the complete figure, under the middle of any bay, represent the maximum shear in that bay; and also (§ 41) the reaction at the abutment A when the portion of the bridge to the *right* of that bay is loaded with the travelling load. Suppose the girder cut across by a section plane $x\,y$ and the right portion removed. The left portion would then be in equilibrium under the action of the reaction of the abutment A, and of three forces acting at the points of section of the three bars 4 6, 4 5, 3 5, and equal and opposite to the stresses in those bars. It is required therefore to resolve a force whose magnitude is represented by $c\,d$ (Fig. 75A) and which acts at A, along three given directions.

Produce 6 4 to cut the vertical at A in e: join e 5. For convenience, $c\,d$ is transferred to $c'\,d'$. Draw c' 4′ parallel to 6 4, and d' 4′ parallel to e 5, 4′ 5′ parallel to 4 5, and d' 5′ parallel to 5 3. Then (§ 11) 4′ 5′ is the component of the reaction acting along 4 5, and 4′ 5′ is therefore the maximum stress in 4 5. The determination of the stresses in the other diagonals 2 3, 6 7, 8 9 is similarly carried out; these stresses are all tensile. The stresses in the dotted bars will, by symmetry, be respectively similar. Thus, the maximum in 1 4 is equal to that in 2 3, and so on.

These stresses may also be determined by taking moments about the intersection of the segments of the boom produced. Thus, if p is the stress in 4 5, and o the intersection of the bars 6 4, 5 3 produced,

$$p \,.\, \mathrm{L} = cd \,.\, r;$$

or,

$$p = cd \,.\, \frac{r}{\mathrm{L}}.$$

The lengths L and r can be taken from the drawing, and the value of p thence deduced. The great distance of the point o will, however, often render this method impracticable.

Stresses in Verticals.—Suppose the girder cut by a plane $z\,z$, and resolve the maximum shearing stress $c\,d$ in the bay 3 5, acting as a reaction at A along the three bars 2 4, 4 3, 3 5. The construction is carried out as before, $c\,d$ being transferred to $c''\,d''$ (Fig. 75b), $4''\,3''$ is obtained as the maximum stress in the vertical 4 3. This stress is a pressure. This construction is carried out in Fig. 75b for the verticals 4 3, 6 5, 8 7.

The principle of moments might be employed here also, the moments being taken about the intersection of the bars 4 2, 5 3 produced.

The stress in the end vertical 2 1 cannot be obtained by either of these methods. By resolving the *maximum* reaction at A along the bars A 2, A 1 the stresses in those bars are obtained. Compounding the stress in A 1 with the load at 1 (10·125 tons), and with the stress in 1 4, the stresses in 2 1 and 1 3 are determined.

Stress in Booms.—By combining the stress diagrams for the dead load with that of the travelling load covering the whole girder, the stress diagram for the girder when fully loaded is obtained. The ordinates of this diagram under the centre of each bay can then be severally resolved along the bars in that bay as above, and the maximum stresses in the booms are arrived at.

The stresses in the booms can also be derived from the funicular polygon of total load, or from the curve of maximum bending moment.

On a load line $l\,l'$ (Fig. 75d) set off successively the total loads on the pairs of joints 2 1, 4 3, &c. (these total loads are equal to 10·8 tons). Take O as pole, making l' O equal to a convenient distance. Draw half of the funicular polygon of these loads as shown (Fig. 75c). Then the bending moment at 5 (Fig. 75) is equal to $m\,n$. H, where H = O l'. From 5 draw a perpendicular $5\,t$ on 4 6 (Fig. 74), then, if Q is the maximum stress in 4 6,

$$Q \,.\, 5t = mn \,.\, H;$$

or,

$$Q = mn \,.\, \frac{H}{5t};$$

whence Q can be obtained by calculation or construction. By drawing a perpendicular from 6 on 3 5 the maximum stress in 3 5 can be determined; in this case, however, by symmetry the compressive stress in 4 6 is equal to the tensile stress in 3 5.

As has been stated, the stresses in the *end* segments of the booms can be obtained by a resolution of the maximum reaction at the abutment along the directions of those segments.

The maximum stresses in the booms may also be obtained by drawing a reciprocal diagram of the whole girder, according to the method of Chapter IV.

If, as is occasionally the case, the cross girders bearing the platform are attached to the verticals at points intermediate between the upper and lower ends of the latter, the load may be distributed at the upper and lower joints in the ratio of the upper and lower portions into which the verticals are divided by the points of attachment.

61. **Bowstring Girders.**—Fig. 76, Pl. VIII., represents the frame diagram of a bowstring girder of 192′ 0″ span, divided into sixteen bays of 12′ 0″. The whole of the loading is taken to be uniformly distributed, the actual concentrated loads being supposed replaced by an equivalent distributed load. The travelling load on the girder has been taken at 1·0 ton, and the dead load at 0·65 ton per foot, the latter being assumed to act half at the upper and half at the lower joints.

The loading will therefore be—

	Tons.
Dead load on upper joints	3·9
" lower "	3·9
Travelling load on lower joints	12·0
Total " upper "	3·9
" " lower "	15·9
" " end "	9·9

The verticals are intended to act both as struts and as ties, the crossed diagonals as ties only.

The stresses due to dead load, and also the stresses in the booms due to both dead and travelling loads, can be obtained by means of reciprocal diagrams. The latter will somewhat resemble that given in Fig. 46A, the difference being, that in the present instance the lower boom is straight instead of polygonal. In drawing these reciprocal diagrams it will be necessary, as in § 32, to deal with that system of triangles only in which the diagonals are in tension.

Diagonals. Travelling Load.—The load is supposed to come on the girder at B, Fig. 76, Pl. VIII., the diagonals shown in continuous lines will then be in tension, and the maximum stress in any diagonal ef will arise when the travelling load reaches s_7, the middle point of the bay under ef; i. e. when the shearing stress at s_7 is a maximum. Suppose the girder cut by a section plane $\alpha\beta$, and, considering the maximum shearing stress at s_7 to act as a reaction at A, resolve this shearing stress along the directions of the three bars cut by $\alpha\beta$. The component along ef is the maximum stress in ef due to live load, and similarly for the remaining diagonals.

The maxima shearing stresses at the middle points of the several bays of the girder are found by the construction of § 52 (Fig. 70). The load line ba' (Fig. 76a) is made equal to the travelling load (192 tons), and is bisected at right angles by O c; O c is made equal to 100 tons on the scale of loads: at is drawn parallel to O a', b O being produced to cut at in t. A s_7, s_7 B are bisected, and perpendiculars are dropped from their points of bisection to cut at and bt in p and r, then (§ 52) rt is a tangent to the bending moment curve at a point on the latter vertically below s_7. Produce rp to cut the vertical through A in 7: join 7 *b*, and draw O 7′ parallel to 7 *b*. Then if *b* 7″ the total load covering the length B s_7 of the girder, 7′ 7″ is the maximum shearing stress at s_7 (§ 52). The maximum shearing stresses at the points $s_2 \ldots \ldots s_8$ are similarly obtained.

A stress represented by 7′ 7″ (Fig. 76a) is now supposed to act as a reaction at A (Fig. 76), and must be resolved along the bars cut by the section plane $\alpha\beta$. Produce the cut segment of the upper boom to meet the vertical through A in *g*. Draw *l* 7 (Fig. 76b) vertical and equal to 7′ 7″ (Fig. 76b) and *l d* horizontal: draw 7 *n*, *l n* parallel to *h g*, *g f* respectively, and *n d* parallel to *e f*; then (§ 11) *n d* represents the component of 7′ 7″ resolved along *e f*, and *n d* is therefore the maximum stress in *e f* due to travelling load. This construction is carried out in Fig. 76b as far as the eighth diagonal from the left abutment.

Verticals.—The mode of determining the stresses in the verticals is very similar. Suppose the girder to be cut by a plane $\gamma\delta$ (Fig. 76), and resolve the maximum shearing stress at s_7 along the bars cut. In Fig. 76c make *l* 7 equal to 7′ 7″, and draw 7 *v*, *l v* parallel to *e g*′, *g*′ *f* (Fig. 76), then the vertical *v k* gives the component of 7′ 7″ alone *e i*, and therefore the maximum stress in *e i*. The construction is carried out in Fig. 76c for all verticals from the second to the seventh inclusive. The first vertical does not, properly speaking, come into the system of triangulation under consideration; it will be sufficient to make this vertical able to sustain the load on one bay of the girders or the heaviest concentrated axle load if the latter is greater. The dotted diagonals will be similarly strained when the load comes on the girder at A. The above construction can be employed to determine the stresses due to dead load if a shearing stress diagram (§ 46) for dead load is first drawn, and also the stresses in the booms due to travelling load if a shearing stress diagram of travelling load covering the girder (§ 52) is drawn. The stresses in the booms may also be obtained from the curve of bending moments.

The stresses due to dead and travelling load should in all cases be carefully tabulated and then added, having due regard to their sign.

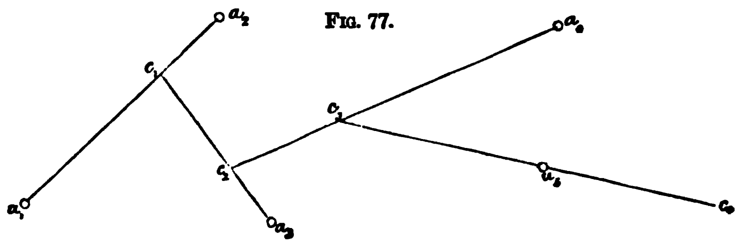
CHAPTER VIII.

CENTRE OF PARALLEL FORCES.—CENTRE OF GRAVITY OF PLANE FIGURES.

62. Parallel Forces acting at Points in one Plane.—If a system of parallel forces p_1; $p_2 \ldots$ acting at any points a_1; $a_2 \ldots$ (Fig. 77) in the plane of the paper are supposed to turn about those points still retaining their parallelism, then the resultant of the whole system will always pass through a fixed point, in the same plane with a_1; a_2..., termed the *centre of the parallel forces.*

This centre can be found as follows: join $a_1 a_2$ (Fig. 77), and divide $a_1 a_2$ in c_1, such that $\frac{a_1 c_1}{c_1 a_2} = \frac{p_2}{p_1}$: then (§ 13) c_1 is the *centre* of p_1 and p_2. Suppose $p_1 + p_2$ to act at c_1 and join $c_1 a_3$. Divide $c_1 a_3$ so that $\frac{c_1 c_2}{c_2 a_3} = \frac{p_3}{p_1 + p_2}$. Then c_2 is the centre of p_1, p_2, and p_3, and by carrying on the process till the last force has been dealt with, the centre of the

FIG. 77.

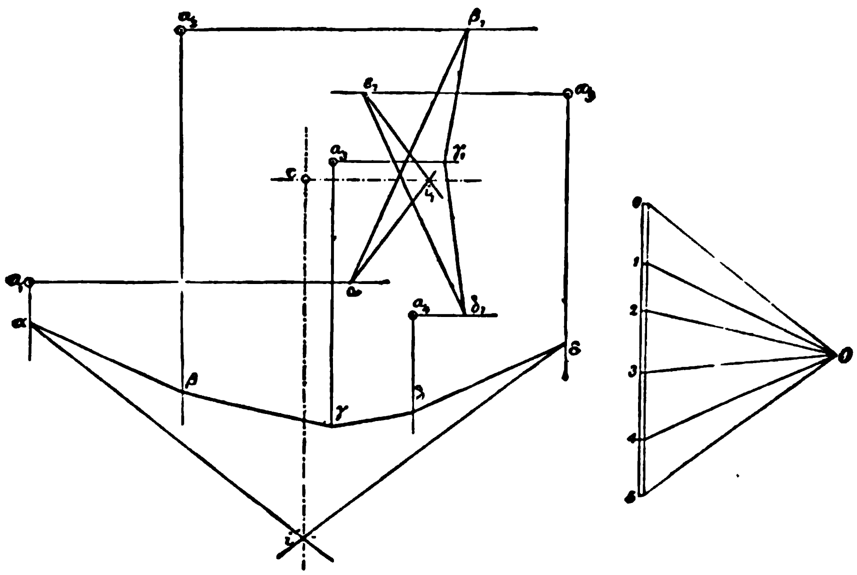


system is obtained. If any one of the forces; *e g*, p_5 is of opposite sense to the rest, then c_4 will lie on $c_3 a_5$ produced, so that $\frac{c_3 c_4}{a_5 c_4} = \frac{p_5}{p_1 + p_2 + p_3 + p_4}$. The production must always be made from the point of application of the less in the direction of the point of application of the greater of the two forces.

The above method would be tedious if the number of forces were great, and in this case the centre can be more readily found by the employment of the funicular polygon. Since the parallel forces may be supposed to act in any direction, provided that their parallelism is preserved, they may be supposed to act in the plane of the paper. Through $a_1 \ldots a_5$ (Fig. 78) draw the parallel lines of action in any direction. Parallel to this direction draw a load line $o\,5$, and set off successively on $o\,5$ lengths proportional to the magnitudes of the forces $p_1 \ldots p_5$ acting at $a_1 \ldots a_5$. Take any pole O, and draw $\alpha \ldots \epsilon$, the funicular polygon of the forces. The last sides of this polygon intersect in i, then (§ 12) the resultant of the forces passes through i, and consequently a line through i parallel to the assumed direction of the forces will contain the "centre" of the forces. By drawing the lines of action of the forces through $a_1 \ldots a_5$ in any other assumed direction, and repeating the above construction, a second line will be found containing the centre. It will be simpler to draw these new lines of action at right angles to the previously assumed direction, and then it is not necessary to draw a new polygon of forces, but merely to draw a new funicular polygon, the sides of which are respectively at right angles to those of the first. The funicular polygon $\alpha_1 \ldots \epsilon_1$ is thus drawn, and its two last sides intersect in i_1. Hence c is the centre of the whole system of forces.

It will thus in general be necessary to draw one polygon of forces and two funicular polygons in order to obtain the centre of a system of parallel forces whose points of application lie in one plane. If, however, the forces were symmetrical, and their points of application symmetrically situated with respect to any line, or if all the

FIG. 78.



points of application were on a line, this line must contain their centre, and only one polygon of forces and funicular polygon are needed.

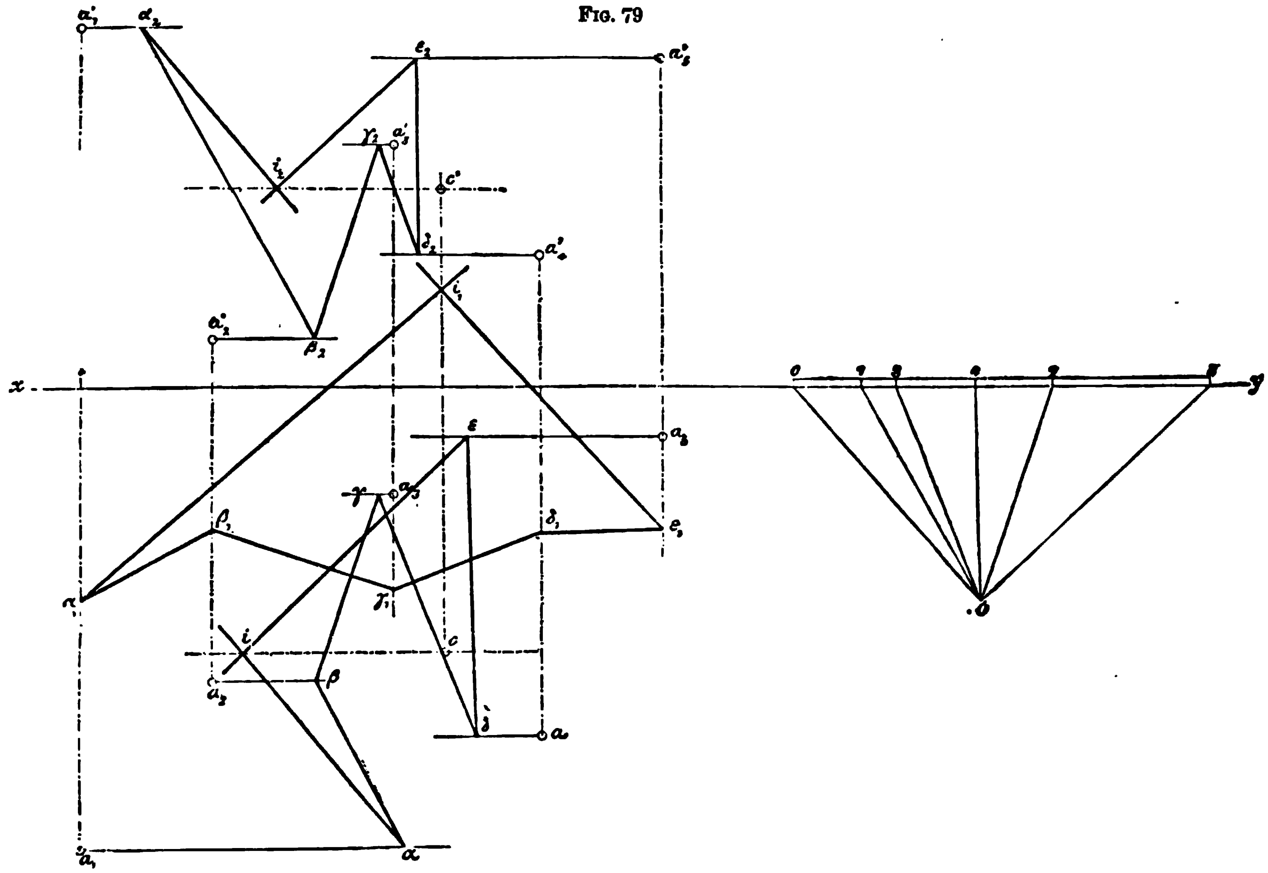
63. **Parallel Forces in Space.**—If the points of application of a system of parallel forces are not in one plane, the forces must be projected on to two co-ordinate planes which it is convenient to take at right angles to each other. In Fig. 79 $a_1 a_1'$; $a_2 a_2'$... are the orthographic *projections* of the points of application of a system of parallel forces $p_1, p_2, \ldots$; xy is the ground line of the planes on which the forces are projected. Dealing with the *plans* $a_1, a_2, \ldots$ of the points of application of the forces precisely as in the preceding section, by means of one polygon of forces and two funicular polygons, a point c is obtained, which is the *plan* of the required centre. Dealing with $a_1', a_2'\ldots$, the *elevations* of the points, by means of a funicular polygon, a line is obtained containing the *elevation* of the centre. By projecting up from the plan c on to this line the elevation c' of the centre is determined.

Thus, set off the forces $p_1, p_2, \ldots$ in succession along a load line 0 5, which load line may conveniently be taken on xy, the forces being supposed temporarily to act in directions parallel to the vertical plane, so that the plans of their lines of action drawn through $a_1 a_2 \ldots$ will be parallel to xy. Draw the funicular polygon $\alpha \ldots \epsilon$ of the forces, then a line parallel to xy drawn through the intersection i of the first and last sides of this funicular polygon will contain the plan of the centre. Suppose the parallel forces to act at right angles to the vertical plane, so that their plans are at right angles to xy, and draw a second funicular polygon $\alpha_1 \ldots \epsilon_1$ whose sides are respectively perpendicular to those of the first.

Through i_1, the intersection of the first and last sides of this second funicular polygon, draw a line perpendicular to xy and cutting the line through i in c. Then c is the *plan* of the required centre of the system.

Dealing with the elevations a_1', a_2', ... of the points of application of the forces,

FIG. 79



suppose the later to act in directions parallel to the horizontal plane and draw their elevations parallel to xy. Draw a third funicular polygon $\alpha_2 \ldots \epsilon_2$ whose sides are respectively parallel to those of the first funicular polygon $\alpha \ldots \epsilon$, then a line parallel to xy through i_2, the intersection of the first and last sides of this funicular polygon will contain the elevation of the centre of the system; projecting from c on to this line, we obtain c', and the required position of the centre of the parallel forces is therefore determined.

The magnitude of the resultant of the system is equal to Σp, the algebraic sum of the forces.

In the two preceding sections the forces have been all supposed to have the same sense, if any of them are of opposite sense, the fact must be remembered when the forces are being set off along the load line. The forces forming a system may be split up into groups, if then the *centre* of each group is found, and the resultant of the group is supposed to act as a single force at that centre, the centre of the whole system can be found by dealing with these resultants acting as parallel forces at the centres of groups by means of the preceding section.

64. **Centre of Gravity of Lines and Curves.**—If the forces in the preceding sections are supposed to be replaced by the *weights* of a system of bodies, or of the several portions of the same body—which weights form a system of parallel, because all vertical, forces—then the centre of the parallel forces is termed the centre of gravity of the system of weights, or of the body. The centre of gravity of a body may therefore be defined as the point through which the resultant of the weights of the particles of the body acts in whatever position the body is placed, or as "the point at which the whole weight of the body may be supposed to be concentrated without altering its statical effect." In applying the term to lines or plane figures, the former must be conceived as made up of heavy points, and the latter of heavy lines.

Broken Line.—The centre of gravity of the several segments of the broken line will be at their respective middle points. Suppose parallel forces proportional to the lengths of the segments to act at their middle points, and find their centre as in § 62 if the line lies in a plane, or § 63 if otherwise; this is the centre of gravity of the line. If the segments are all equal and make equal angles with each other so that the broken line is part of a regular polygon, as in Fig. 80, the centre of gravity can be found in a much shorter way as follows: Join af, and bisect af by a perpendicular: this perpendicular is evidently a line of symmetry, and contains the centre of gravity of the line, as well as O the centre of the circle passing through $a \ldots . f$. Through O draw any line xy. Then if l is the length of ab, $bc \ldots .$; $x_1 x_2 \ldots$ the perpendicular distances of their respective middle points from xy; L the whole length of the line, and X the perpendicular distance of its centre of gravity from xy—

$$\Sigma(l \,.\, x) = \mathrm{L} \,.\, \mathrm{X}.$$

But if $c' d'$ is the projection on xy of any segment cd—

$$\frac{cd}{c'd'} = \frac{r}{x_2}; \text{ or, } cd \,.\, x_2 = c'd' \,.\, r$$

where r is the radius of the circle through $a \ldots e$.

Hence—

$$\mathrm{L} \,.\, \mathrm{X} = r \; (c'd')$$

and—

$$\mathrm{L} \,.\, \mathrm{X} = r \,.\, a'f'.$$

Construct a right-angled triangle $f'n'n$ whose hypothenuse $f'n$ is equal to L

N

and perpendicular $n'n$ equal to r. With f' as centre, and $f'a'$ as radius, describe a circle cutting $f'n$ in s; draw $s s'$ perpendicular to $x y$; then, $s s' = \frac{r \cdot a'f'}{L} = X$; draw s G parallel to $x y$, cutting the line of symmetry O k in G. Then G is the centre of gravity of the whole line $a \ldots e$.

FIG. 80.

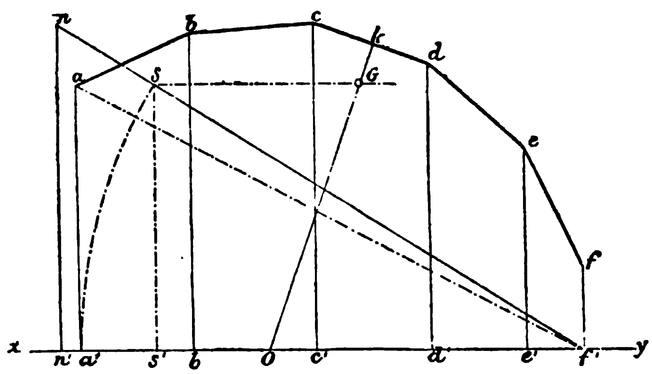
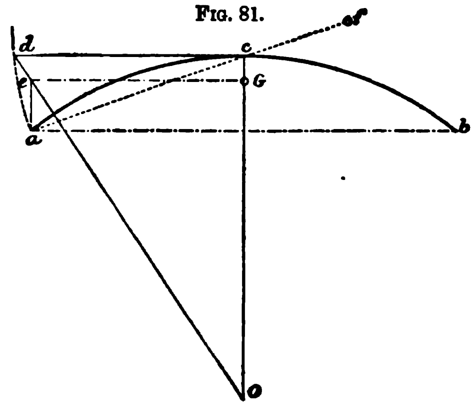


FIG. 81.



Circular Arc.—The construction of the preceding section can be applied to the circular arc $a b$, Fig. 81, since the latter can be regarded as a polygon whose sides are infinitely small.

The centre of gravity of the arc lies on O c the line of symmetry, O being the centre of the arc, and c its middle point; draw the tangent at c and make $c d$ equal to the arc $a c$; join O d, draw $a e$ parallel to O c, and e G parallel to $a b$, then G is the required centre of gravity.

In order to set off the length of the arc $a c$ approximately along the tangent at c; join $a c$ and produce $a c$ to f, making $c f$ equal to half $a c$. With f as centre, and $f a$ as radius, describe an arc cutting the tangent in d. Then $d c$ is approximately equal to the arc $a c$.

The centre of gravity of *any* curved line can be found by cutting it up into small segments and treating these segments as straight lines.

65. **Centre of Gravity of Plane Figures.**—The centre of gravity of a triangle is found at the point of intersection of lines drawn from any two angles bisecting the opposite sides; this point is at a distance of $\frac{1}{3}$ of either of these lines, measured along them from the point of bisection.

Parallelogram.—The centre of gravity is at the intersection of the diagonals.

Trapezium.—The centre of gravity must lie in the line of symmetry $e f$ (Fig. 82) bisecting the parallel sides $b c$, $a d$. Draw the diagonal $b d$ dividing the trapezium into two triangles, and obtain $g_1 g_2$ the centres of gravity of these triangles. Join $g_1 g_2$, cutting $e f$ in G, then G is the centre of gravity of the trapezium.

If $bc = l$ and $ad = L$, then it can be shown that G divides ef in the proportion

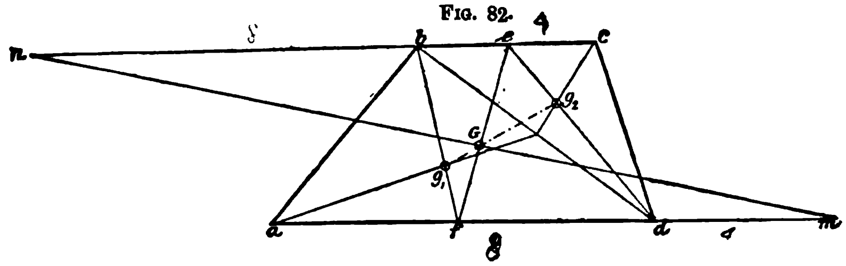
$$l + 2L : 2l + L;$$

or,

$$\frac{l}{2} + L : \frac{L}{2} + l.$$

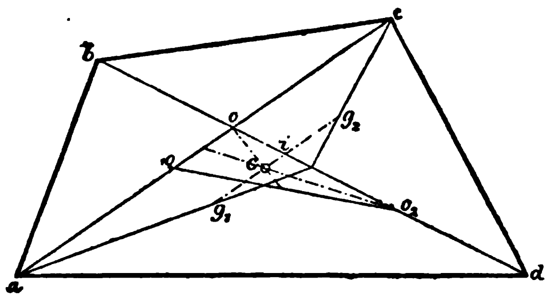
Produce cb, ad in opposite directions to n and m, making $dm = bc = l$, and $bn = ad = L$. Join mn, then mn cuts ef in G.

FIG. 82.



Four-sided Figure.—Draw the diagonals ac, bd of the four-sided figure $abcd$ (Fig. 83), find g_1, g_2 the centres of gravity of the two triangles into which the figure is divided by the diagonal bd: join $g_1 g_2$, then the centre of gravity of the whole figure lies on $g_1 g_2$, and if the centres of gravity of the two other triangles in which the figure is divided by the diagonal ac are obtained and joined, the intersection of the lines joining these pairs of centres of gravity will be the centre of gravity G of the whole figure. The point G divides the distance $g_1 g_2$ in the inverse proportion of the areas of the triangles bad and bcd. If i is the point where $g_1 g_2$ cuts bd, it can easily be shown that $g_2 i$ is equal to g_1 G. It can further be proved that if o is the intersection of the diagonals and $a o_1$, $d o_2$ are equal to $c o$, $b o$ respectively, then the centre of gravity of the whole figure coincides with that of the triangle $o\, o_1\, o_2$.

FIG. 83.



Polygons.—The centre of gravity of any polygon can be found by dividing it into four-sided figures or triangles, and finding the centre of gravity of each. At the centre of gravity of each portion a force proportional to the area of that portion must be supposed to act; then, by means of a polygon of forces and two funicular polygons, the centre of the forces or the centre of gravity of the whole polygon can be found as in § 62. To obtain lines whose lengths are proportional to the areas of the triangles into which the polygon is split up, the construction of § 6 may be employed. The following simple construction can also be adopted. Draw two lines Ox Oy (Fig. 84) at any convenient angle. Suppose H, B; H_1, B_1; H_2, B_2...to be the heights and bases of the triangles to be dealt with. It is merely necessary to reduce these triangles to any common base. Taking B as this base, H represents the areas of the 1st triangle. Set off O b (= B) and O $_1$ (= B_1) along O y, and O h_1 (= H_1) along O x. Join b h_1, and draw b_1 x_1 parallel to h h_1. Then

$$\frac{Ox_1}{Ob_1} = \frac{Oh_1}{Ob}; \text{ or,}$$
$$Ox_1 . B = H_1 . B_1 .$$

Hence O x_1 represents the area of the 2nd triangle at the base B. Any number of triangles, rectangles, or parallelograms can be similarly reduced to a common base.

FIG. 84.

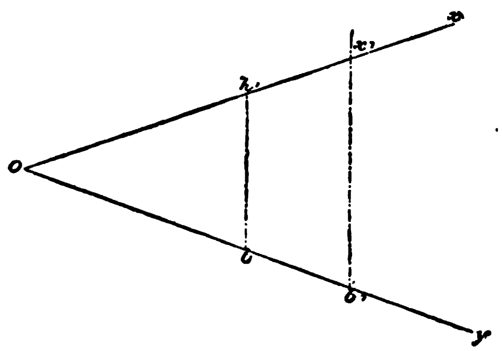
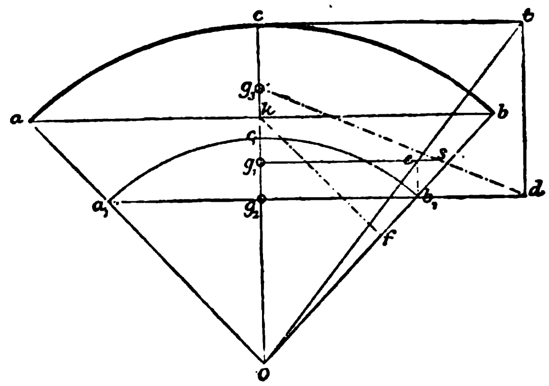


FIG. 85.



Circular Sector.—The sector $a\,c\,b$ O (Fig. 85) may be supposed made up of a system of triangles whose common vertex is at O, and whose bases, infinitely small, are on the arc $a\,c\,b$. The arc $a_1\,c_1\,b_1$, struck with a radius equal to $\frac{2}{3}$ of a O, contains the centres of gravity of all these triangles, the weights of the latter can therefore be supposed to act in the arc $a_1\,c_1\,b_1$, and the centre of gravity of this arc is the centre of gravity of the sector. The construction above described can be employed: draw $c\,t$ tangent to the arc $a\,c\,b$ at c, and equal to the arc $c\,b$: join t O, cutting a line through b_1 at right angles to $a_1\,b_1$ in e, draw $e\,g_1$ through e parallel to $b_1\,a_1$, and cutting the line of symmetry O c in g_1, then g_1 is the centre of gravity of the arc $a_1\,c_1\,b_1$, and of the sector $a\,c\,b$ O.

Circular Segment.—The segment $a\,b\,c$ (Fig. 85) is equal to the difference between the sector $a\,c\,b$ O and the triangle $a\,b$ O. Join $_1b_1$, cutting O c in g_2, then g_2 is the centre of gravity of the triangle. Supposing then that weights proportional to the areas of the sector and triangle act at g_1 and g_2 respectively, it is necessary to find a point g_3 on O c such that

$$\frac{g_1\,g_3}{g_2\,g_3} = \frac{\text{area of triangle}}{\text{area of sector}} = \frac{k\,f}{\text{arc } c\,b} = \frac{k\,f}{c\,t},$$

where $k\,f$ is the perpendicular from k, the middle point of $a\,b$, on O b.

Produce $g_2 b_1$ to d, making $g_2 d$ equal to $c\,t$. Draw $g_1 s$ parallel to $a\,b$ and equal to $k f$. Join $d\,s$, and produce $d\,s$ to cut O c in g_3; then g_3 is the centre of gravity of the segment $a\,c\,b$.

Annular Segment.—The annular segment $a\,c\,b\,e\,f\,d$ (Fig. 86) is the difference between the two circular sectors $a\,c\,b$ O and $d\,f\,e$ O, determine first the centres of gravity $g_1\,g_2$ of these sectors respectively, and then obtain a point G, such that

$$\frac{g_1 G}{g_2 G} = \frac{\text{area of } a\,c\,b\,O}{\text{area of } d\,f\,e\,O} = \frac{(a\,O)^2}{(O\,d)^2}.$$

Join $a\,f$, and draw $d\,f'$ parallel to $a\,f$.

Then $$\frac{O\,f'}{O\,f} = \frac{O\,d}{O\,a};\ \text{or, } O\,f' = \frac{(O\,d^2)}{O\,a}.$$

Draw $g_1\,i$ and $g_2\,k$ at right angles to c O, and respectively equal to O f' and O a. Join $k\,i$, and produce it to cut c O in G. Then G is required centre of gravity.

FIG. 86.

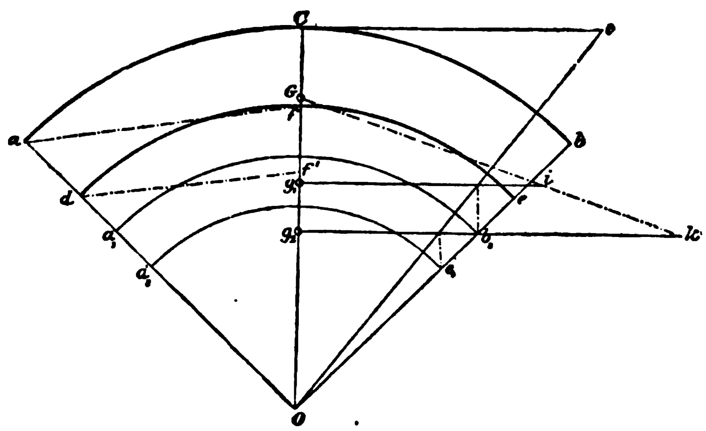
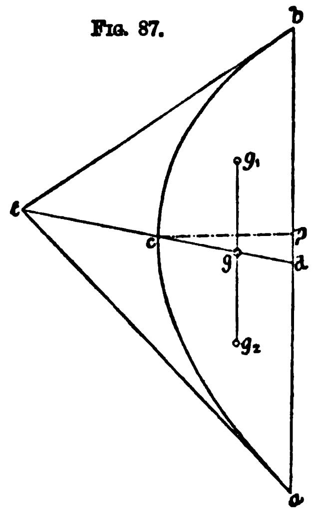


FIG. 87.



Parabolic Segment.—The centre of gravity g of the parabolic segment $b\,c\,d$ (Fig. 87) is on $d\,c$, the line which bisects $b\,a$ and all other parallel chords, and $g\,d = \frac{2}{5} \cdot d\,c$.

The centres of gravity of the half segments $b\,c\,d$, $a\,c\,d$ are situated at points $g_1\,g_2$ on a line through g parallel to $a\,b$, and

$$g\,g_1 = g\,g = \tfrac{3}{8}\,b\,d.$$

The area included between the arc $a\,c\,b$, and the tangents $a\,t$, $b\,t$ at a and b respectively, is the difference between that of the triangle $a\,t\,b$ and the segment $a\,c\,b$. The area of this segment is $\frac{2}{3}\,c\,p\,.\,a\,b$ where $c\,p$ is the perpendicular from c on $a\,b$. The centre of gravity of the area $a\,t\,b\,c$ can therefore be found by determining the centres of gravity of the triangle and of the area $a\,t\,b\,c$, and proceeding as in previous cases. The triangle and the rectangle $\frac{2}{3}\,c\,p\,.\,a\,b$ must be reduced to a common base (§ 6), in order to obtain the lengths of lines proportional to their areas.

Irregular Figures.—The centre of gravity of such a figure as the section of a railway bar can be found by cutting up the section into strips by lines perpendicular to its axis of symmetry. These strips will be very approximately rectangles or trapeziums whose areas, if their breadths are the same, will be proportional to their mean heights, or if the breadths vary they must be reduced (Fig. 84) to any convenient common base. Forces proportionate to these reduced areas must be supposed to act at their respective centres of gravity in a direction perpendicular to the axis of symmetry, then, by means of the polygon of forces and funicular polygon, a line is obtained (§ 62) which will intersect the axis of symmetry in the required centre of gravity.

If the section, the centre of gravity of which is required, has no axis of symmetry, it must be divided up into simple areas in the most convenient way, forces proportional to the reduced areas of the several portions must be supposed to act at their respective centres of gravity, the *centre* of these forces can then be found by the construction of § 62.

66. **Curved Surfaces and Solids.**—It is not proposed to give any constructions for determining the centres of gravity of curved surfaces and solids, but merely to state their positions for simple forms. If any surface or solid can be split up into portions such that the area, or volume and the position of the centre of gravity of each portion is known, then the centre of gravity of the whole surface or solid can be obtained by supposing parallel forces proportional to the areas or volumes of the several portions to act at their respective centres of gravity, and determining the centre of these parallel forces as in § 63.

Pyramidal or Conical Surface.—The centre of gravity is on the line joining the centre of gravity of the base to the vertex at a distance of $\frac{1}{3}$ of this line from the base.

Hemispherical Surface.—The centre of gravity bisects the axis of symmetry.

Surface of Hemispherical Segment.—The centre of gravity bisects the axis of symmetry.

Surface of Frustum of Pyramid.—If the frustum is contained between two parallel planes, the centre of gravity divides the line joining the centres of gravity of the two ends in the proportion $a + 2b : 2a + b$

when a and b are the two parallel sides of any one of the trapeziums which form the faces; or, $P + 2p : 2P + p$

when P and p are the perimeters of the polygons which form the ends. In either case the *smallest* portion of the divided line must of course be that nearest to the *largest* end of the frustum.

Surface of Frustum of Cone.—The centre of gravity divides the line joining the centres of gravity of the two ends in the proportion

$$D + 2d : 2D + d$$

when D and d are the diameters of the two ends, if circular.

Pyramid.—The centre of gravity is on the line joining the centre of gravity of the base with the vertex, and is at a distance of $\frac{1}{4}$ of the whole line from the base.

The centre of gravity of a triangular pyramid coincides with that of four equal weights placed at its angular points.

Cone.—Same as the pyramid.

Hemisphere.—The centre of gravity is at a distance of $\frac{3}{8}$ of the radius, measured from the centre of the base up the axis of symmetry.

Frustum of Pyramid.—The centre of gravity is on the line joining the centres of gravity of the ends, and if G is the centre of gravity of the frustum, g_1 that of the whole pyramid, and g_2 that of the pyramidal portion cut away, then—

$$\frac{g_2 G}{G g_1} = \frac{V}{v}, \text{ or } \frac{a^3}{b^3}$$

where V and v are the volumes of the whole pyramid and of the portion cut away, and a and b are any homologous sides of those pyramids respectively.

Frustum of a Cone.—Using the same lettering as before—

$$\frac{g_2 G}{G g_1} = \frac{V}{v}, \text{ or } \frac{L^3}{l^3}, \text{ or } \frac{R^3}{r^3}$$

where L and l are the respective slant heights, and R and r the respective radii of the bases of the whole cone and of the conical portion cut off.

Fig. 88

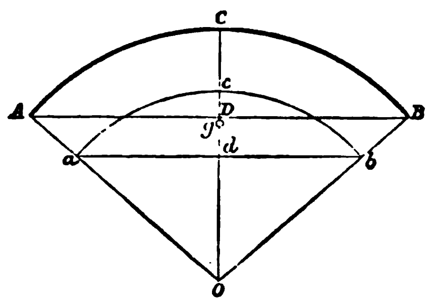
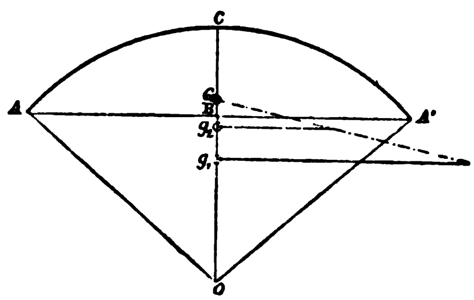


Fig. 89.



Conical Sector.—The centre of gravity of the conical sector generated by the revolution of the sector O A C (Fig. 88) about C O may be found by describing the arc $a c b$ with radius equal to $\frac{3}{4}$ O A, joining $a\,b$ and bisecting $c\,d$ in g: g is the centre of gravity, and the distance g O is equal to $\frac{3}{4}$ O C − $\frac{3}{8}$ C D.

Segment of a Sphere.—The centre of gravity of the spherical segment generated by the segment A C A′ (Fig. 89) revolving about B O is situated at a point G on B C, such that

$$\frac{g_2 G}{g_1 G} = \frac{a}{r}$$

where g_1 is the centre of gravity of the cone O A A′, g_2 the centre of gravity of the conical sector O A C A′, r the radius O A, and

$$a = \frac{(2r - h)(r - h)}{2r}$$

where h is the height B C of the segment: a is readily constructed as a fourth proportional to $2r$; $r - h$; and $2r - h$.

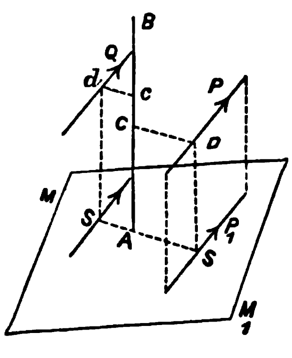
Solids bounded by Irregular Surfaces.—The centres of gravity of such solids may be found by supposing them cut up by parallel planes into a large number of slices. If the thickness of these slices is small, the centre of gravity of each will be very nearly the same as that of its mean section, i.e. its section by a plane parallel to and bisecting the distance between the two planes by which it is bounded. If the thickness of the slices is the same, their volumes will be approximately as the areas of their mean sections. Supposing parallel forces proportional to the volumes of the slices to act at their respective centres of gravity, the centre of these parallel forces can be determined as in § 62, or § 63.

CHAPTER IX.

MOMENT OF INERTIA—CENTRAL ELLIPSE, &c.

67. **Moments of Parallel Forces.**—The moment of a system of co-planar forces about a *point* in their plane has been explained in § 16, as the sum of the products of each force into its perpendicular distance from the point, and is equal to the product of the resultant of the forces into its perpendicular distance from the point. The physical effect of the action of such forces is a tendency to produce rotation about an axis perpendicular to the plane of the forces and meeting it in the given point.

FIG. 89A.

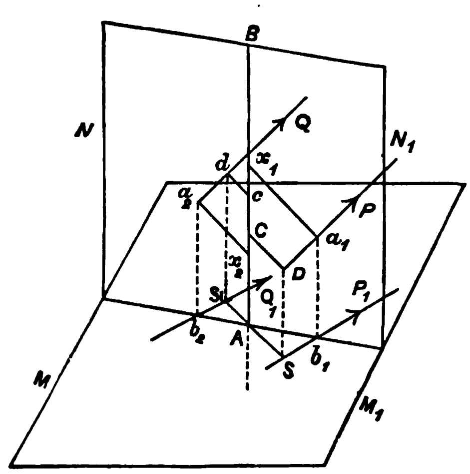


The moment of a force P about an axis A B (Fig. 89A), when the force does not act in a plane perpendicular to the axis, may be defined as the product of the projection, P_1, of the force on a plane M M_1 perpendicular to the axis A B, into the distance A S of the point in which A B intersects the plane, from the projection P_1; in other words, the moment of a force with respect to an axis is the moment of the projection of the force on a plane normal to the axis with respect to the *point* in which the axis intersects the plane of projection.

The distance A S of the foot of the axis from the projection of the force is, moreover, equal to the length of the line C D, which is perpendicular to P and the given axis, i.e. is the shortest distance between them.

A definite physical meaning can be attached to the foregoing, viz.: that the force being resolved into 2 components, one P_1 in a plane perpendicular to A B, and the other parallel to A B, i.e. into P. cos. θ and P. sin. θ, where θ is the angle between P and the plane perpendicular to A B. The former tends to produce a rotation measured by the product $P_1 \times AS$ about A B, and the latter tends to produce a translation along A B. If we have a second force Q parallel to P, its moment about A B is similarly Q_1 or Q cos. θ multiplied by A S_1, A S and A S_1 being in the same straight line. The algebraical sum of the moments of the components P_1 and Q_1 about A B is equal to the moment of their resultant, i.e. $P_1 \times AS - Q_1 \times AS_1$ = moment of R_1 about A B, where R_1 is the resultant of P_1 and Q_1; but $R_1 = R \cos. \theta$, where R is the resultant of P and Q. Hence, the moment of two parallel forces about any axis is *proportional to* the product of their resultant into the shortest distance between it and the given axis. This same is evidently true for any number of parallel forces.

FIG. 89B.



Draw any plane as N N_1 (Fig. 89B) through A B and let the line of action of P meet it in a_1 and that of Q meet it in a_2, and let the line of intersection of the two planes M M_1, N N_1 meet P_1 in b_1 and Q_1 in b_2.

Then the algebraical sum of the moments of P and Q about A B is proportional to $P_1 \times A b_1 \pm Q_1 \times A b_2$ and also to the projection of their resultant on the plane M M_1 multiplied by the distance between A and the point in which such projection intersects the line $b_1 b_2$. ($= x$ suppose). The equation $P \times A b_1 \pm Q \times A b_2 = R \,.\, x$ is, therefore, evidently true.

Lastly, if from a_1 and a_2 *any* parallel lines are drawn in the plane N N_1 meeting the line A B in x_1 and x_2 respectively; then, since $a_1 x_1 = A b_1$ multiplied by some constant, and $a_2 x_2 = A b_2$ multiplied by the same constant, it follows that the moments of P and Q about A B are proportional to $P \times a_1 x_1$ and to $Q \times a_2 x_2$, and that $P \times a_1 x_1 \pm Q \times a_2 x_2 = R \times$ the line drawn from the point in which R intersects the plane N N_1 parallel to $a_1 x_1$ (or $a_2 x_2$) to meet A B.

No direct physical meaning can be attached to such an expression as $P \times a_1 x_1 \pm Q \times a_2 x_2 \pm \ldots.$ but it nevertheless leads to useful results, and therefore the moment of a system of parallel forces about any axis is sometimes defined as the *algebraical sum of the products of the forces into the distances, measured in any parallel direction, of the points in which their directions meet any plane through the axis, from that axis.*

Thus if the forces $p_1 \ldots p_6$ act at $a_1 \ldots a_6$ (Fig. 90) respectively, their moments about an axis X X lying in the plane containing $a_1 \ldots a_6$ are equal to $p_1 . y_1$, $p_2 . y_2$, ..., where $y_1, y_2 \ldots$ are the distances of $a_1\, a_2 \ldots$ from X X measured in directions making any angle θ with X X.

The total moment of the systems of forces is $\Sigma\ (p\ .\ y)$, and

$$\Sigma (p\ .\ y) = \Sigma p\ .\mathrm{Y}\ ,$$

where Y is the distance of the centre of the system from X X measured in a direction parallel to the distances $y_1,\ y_2 \ldots$ If the points of application $a_1,\ a_2 \ldots$ move on parallel lines, the centre of the system will move on a parallel line. Further, the sum of the moments of a system of parallel forces about any axis passing through their centre is *nil*, and conversely if the sum of the moments about any axis is *nil*, the axis must pass through the centre of the system.

If the sum of a system of parallel forces is zero, then any force must be equal and opposite to the resultant of the remainder, and will form with that resultant a couple. Such a system can be formed into as many couples as there are groups into which it can be split up. If the centres of a pair of groups coincide, the couple reduces itself to two equal and opposite forces acting at a point.

The moment of a force with respect to a plane has no physical meaning unless the plane is parallel to the direction of the force. In that case the moment of the force is the product of the force into the perpendicular distance between it and the plane. Analogously, however, to the above convention as to the moment of a system of parallel forces about an axis, if a plane cuts a system of parallel forces in $a_1, a_2 \ldots\ldots$ and parallel lines are drawn in any direction through $a_1, a_2 \ldots$, &c., meeting a second plane in $b_1,\ b_2 \ldots$, then the sum of the products of each force by the corresponding distance, i.e. $\Sigma\ (p\ .\ a\,b)$, is called the moment of the system about such second plane, and is equal to R . Y where R is the resultant of the forces and Y is the distance measured parallel to $a_1 b_1$, &c., between the point in which R intersects the plane $a_1,\ a_2,\ a_3$, &c., and the plane with respect to which the moments are taken, i.e. the plane $b_1,\ b_2,\ b_3 \ldots$, &c.

If the lines $a_1 b_1$, &c., are all drawn in the plane $a_1,\ a_2,\ a_3$, the moment about the plane is evidently identical with the moment about the line of intersection of the two planes, i.e. with the moment about an axis.

It should be noticed that the expression thus defined as the moment of a system of parallel forces about a plane is of an entirely different character to the moment of a system of co-planar forces about a point in their plane. The latter depends essentially on the directions of the lines of action of the forces, each of which may be supposed to act at any point of its line of action; the former, on the contrary, is entirely independent of the direction of the lines of action of the forces, and varies with the co-planar points which are selected as the points of application of the forces.

68. **Reduction of Moments to a Common Base.** — The reduction to a common base of the moments of a system of parallel forces about a *point* in their plane has already been treated in § 20; a similar construction serves for the reduction of moments about an *axis* parallel to the direction of the forces. In Fig. 90 set off the forces $p_1, p_2 \ldots$ acting at $a_1, a_2 \ldots$ along a load line $l\,d$ parallel to X X, the axis about which moments are to be taken. Through $a_1, a_2, \ldots$ draw the lines of action of the forces parallel to X X. Draw a funicular polygon of the forces with respect to any pole O, and if necessary produce the sides of this funicular polygon to cut X X in $b_0, b_1, \ldots b_5$ successively. From O, draw O S parallel to the directions of $y_1, y_2 \ldots$; then $b_0\,b_1$ represents the moment of p_1 about X X, $b_1\,b_2$ that of p_2, and so on, all these moments being reduced to a common base O S. Moreover, $b_0\,b_5$ represents the reduced moment of the resultant, and $b_0\,b_5$. O S is equal to the sum of the moments of all the forces, or—

$$b_0\,b_5 \,.\, \mathrm{O\,S} = \Sigma(p\,.\,y).$$

If the lever arms are supposed to be perpendicular to the axis, then $\theta = 90°$ and the moment base is O H drawn from O perpendicular to the load line $l\,d$, so that in this case—

$$b_0\,b_5 \,.\, \mathrm{O\,H} = \Sigma\,(p\,y).$$

In future the lever arms will always be supposed perpendicular to the axis.

69. **Moment of Inertia of a System of Parallel Forces.** — If $p_1, p_2 \ldots$ are parallel forces acting at points $a_1, a_2 \ldots$ distant $y_1, y_2 \ldots$ from an axis X X in the same plane, then $\Sigma\,(p\,y^2)$ is the moment of inertia of the system about X X. The *sign* of the moment of inertia is therefore independent of that of the distances $y_1, y_2 \ldots$ and dependent only on the *sense* of the forces.

The moment of inertia can be obtained graphically as follows:—Reduce the moments of the forces to any common base O H (Fig. 90), as in the preceding section, and suppose that parallel forces whose magnitudes are represented by the reduced moments $b_0\,b_1, b_1\,b_2 \ldots$ act parallel to X X at the given points of application; $b_0\,b_1$ at a_1, $b_1\,b_2$ at a_2, and so on. By means of a new funicular polygon, reduce the new system of forces to a common base O′ H′ (Fig. 90), setting off the distances $b_0\,b_1$, $b_1\,b_2 \ldots$ on a new load line also parallel to X X. The sides of the new funicular polygon (produced if necessary) successively cut X X in $c_0, c_1, \ldots$

Then—

$$c_0\,c_1 \,.\, \mathrm{O'\,H'} = b_0\,b_1 \,.\, y_1.$$

But—

$$b_0\,b_1 \,.\, \mathrm{O\,H} = p_1. \text{ [1]}$$

Hence— $p_1 y_1^2 = c_0 c_1 \,.\, \mathrm{OH} \,.\, \mathrm{O'H'}.$

Similarly— $p_2 y_2^2 = c_1 c_2 \,.\, \mathrm{OH} \,.\, \mathrm{O'H'}.$

And— $\Sigma (p y^2) = (c_0 c_1 + c_1 c_2 + \ldots) \mathrm{OH} \,.\, \mathrm{O'H'},$

the *sign* of the intercepts $c_0 c_1$, $c_1 c_2$, ... depending on the direction in which the force tends to produce rotation.

Thus— $\Sigma (p y^2) = c_0 c_5 \,.\, \mathrm{OH} \,.\, \mathrm{O'H'},$

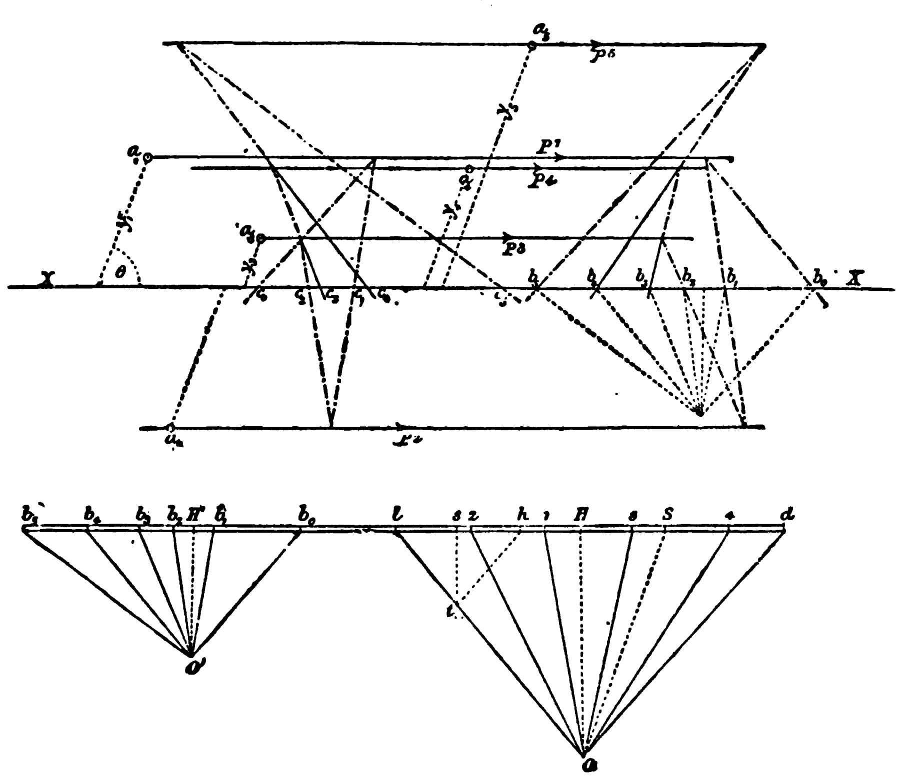
or if O′ H′ has been taken equal to O H—

$$\Sigma (p y^2) = c_0 c_5 \,.\, \overline{\mathrm{OH}}^2.$$

By repeating the above process moments of higher order, such as $\Sigma (p y^3)$, could be dealt with graphically.

In Fig. 90 the intercepts $b_0 b_1$, $b_1 b_2$...have all been transferred to a load line

FIG. 90.



parallel to XX; it will, however, save trouble to use the intercepts themselves as forming the load line: this is shown by the dotted vectors. Similarly it will usually be convenient to take the load line *on* the axis about which moments are to be taken.

It will be noticed that two of the sides of the second funicular polygon accidentally cross on the line of action of p_3.

It is evident that the second system of forces—i.e. the reduced moments of the forces supposed to act at their respective given points of application— will have a "centre," which could be found by § 62. This centre is termed the *centre of the reduced moments.*

The moment of inertia of a system of parallel forces is of two dimensions in space and one in mass, and is said to be in square foot pounds, or square inch pounds, according as the foot or inch is the linear unit.

70. **Radius of Gyration.**—If $\Sigma(py^2)$ is the moment of inertia of a system of parallel forces p_1, p_2...about an axis X X, and k is a linear magnitude such that $k^2.\ \Sigma(p) = \Sigma(py^2)$, then k is the radius of gyration of the system about the axis X X.

It has been shown in the preceding section that, for the system of parallel forces indicated in Fig. 90,

$$\Sigma(py^2) = c_0c_5 . \mathrm{OH} . \mathrm{O'H'}.$$

Hence—

$$k^2 = \frac{c_0c_5 . \mathrm{OH} . \mathrm{O'H'}}{\Sigma(p)} = \frac{c_0c_5 . \mathrm{OH} . \mathrm{O'H'}}{ld}.$$

Make $lh = \mathrm{O'H'}$. Draw ht parallel to dO and ts parallel to O H. Then by similar triangles $\frac{ts}{\mathrm{OH}} = \frac{lh}{ld}$; or $ts = \frac{\mathrm{OH} . \mathrm{O'H'}}{ld}$.

Hence—

$$k = \sqrt{c_0c_5 . ts},$$

whence k can be obtained by construction. Or, the second base O′ H′ might have been taken equal to $ld\ [=\Sigma(p)]$, in which case—

$$k = \sqrt{c_0c_5 . \mathrm{OH}}.$$

If K is the radius of gyration of a system of parallel forces about an axis passing through their centre, and k the radius of gyration of the system about a second axis parallel to and distant d from the first, then—

$$k^2 = \mathrm{K}^2 + d^2.$$

Hence, if the radius of gyration about an axis through the centre of the system has been obtained, the radius of gyration about any parallel axis at a known distance from the first can be determined, and *vice versa.*

The radius of gyration determined as above must be regarded as a line making the same angle with the axis as that made by the lever arms of the moments of the forces. Like these lever arms, therefore, it will here be supposed invariably perpendicular to the axis.

71. **Curve of Inertia and Central Curve.**—Let k be the radius of gyration of a system of parallel forces about any axis X X, and suppose two lines to be drawn on either side of X X parallel to and distant k from X X. Then, if the axis X X is

supposed to turn about any point O on its direction, lines drawn parallel to, and at distances from the new positions of the axis respectively equal to the radii of gyration about those new positions, will be tangents to a curve of the second order having O as centre. This curve, which is either an ellipse or an hyperbola, is termed the *Ellipse*, or *Hyperbola of Inertia* of the system of parallel forces. In general, the curve is called the *Curve of Inertia*. This curve of inertia is fixed if its centre O is given or assumed.

If the sign of the moment of inertia $\Sigma\,(p\,y^2)$ is the same as that of Σp, k^2 is positive, and the curve of inertia is an ellipse. If all the forces have the same sense, k^2 is positive and the curve is always an ellipse, and as this is commonly the case in practice, the ellipse of inertia alone will here be dealt with. If there are positions of the axis for which k^2 is negative (i. e. the sign of $\Sigma\,(p\,y^2)$ differs from that of $\Sigma\,(p)$, the curve is an hyperbola. The same constructions will, however, apply in the latter case.

If the point O (the centre of the curve of inertia) is the centre of the system of forces, the curve is termed the *Central Curve*; or, in the general case which arises in practice, the *Central Ellipse*. A system of parallel forces can, therefore, have an infinite number of curves of inertia, but only one central curve.

If the points of application of the forces are not in one plane, the curve of inertia becomes a surface, termed the *Ellipsoid*, or *Hyperboloid of Inertia*, and the central curve becomes the *Central Ellipsoid*, or *Hyperboloid*.

If the ellipse of inertia, or the central ellipse, of a system of forces can be drawn, then the radius of gyration about any axis passing through its centre is readily obtained. Suppose $a\,b'\,a'\,b$ (Fig. 91) to be the ellipse of inertia, or the central ellipse of a system of forces, and that the radius of gyration about any axis Y Y passing through its centre, O is required. Draw tangents to the ellipse parallel to Y Y; then, by the property above enunciated, the distance k of these tangents from Y Y is the radius of gyration about Y Y. Moreover, if the ellipse in question is the central ellipse of a system of forces, the radius of gyration (k) about *any axis whatever* can be obtained. For, from the ellipse itself, the radius of gyration (K) about a parallel axis passing through its centre can be obtained, and $k^2 = K^2 + d^2$, where d is the distance apart of the parallel axes.

The ellipse of inertia and the central ellipse will in general have to be drawn from two conjugate diameters, or from the principal axes. (Two diameters of an ellipse are *conjugate* when either bisects all chords parallel to the other.) Thus, determine the radius of gyration k_1 about any assumed axis $a\,a'$ (Fig. 91), and draw $e\,h$, $f\,g$ parallel to and distant k_1 from $a\,a'$. Then if $b\,b'$ were the direction of the axis conjugate to $a\,a'$, determine k_2 the radius of gyration about $b\,b'$, and draw $e\,f$, $h\,g$ parallel to and distant k_2 from $b\,b'$. Then $e\,h$, $h\,g$, $g\,f$, $f\,e$ are all tangents to the ellipse, and $e\,h\,g\,f$ is its circumscribing parallelogram. To draw the ellipse, divide $a\,e$ and $a\,O$ into any the same number of equal parts, and draw lines from b through the divisions of $a\,O$, and from b' through those of $a\,e$. The lines obtained severally intersect on the curve as shown. The ellipse can be completed by similar procedure. Or, if the lengths of the

Fig. 91.

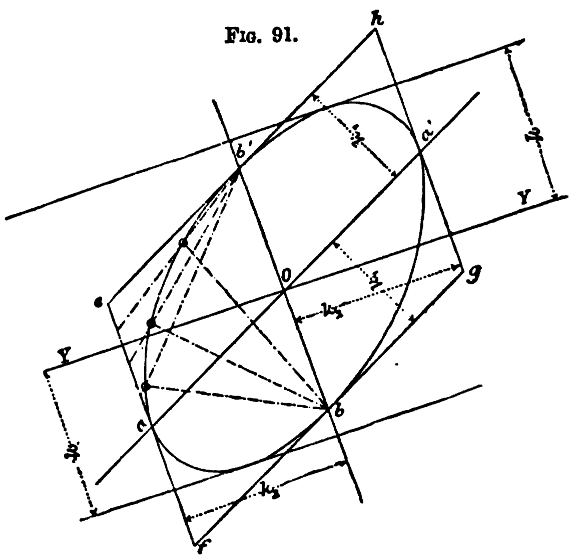
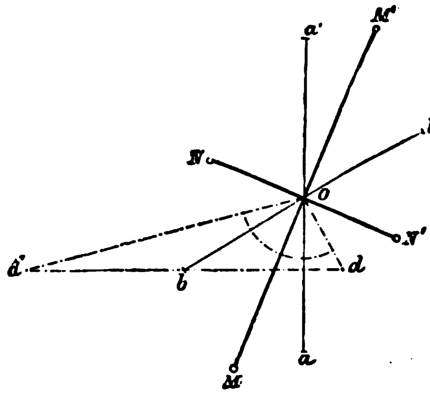


Fig. 92.



conjugate axes $a\,a'$, $b\,b'$ are first determined as above, the principal axes can be obtained directly as follows. Through b (Fig. 92) draw a perpendicular to $a\,a'$, and make $b\,d'$ and $b\,d$ on this perpendicular both equal to $a\,O$. The line bisecting the angle $d'\,O\,d$ is the major axis, and its half-length $M\,O$ is equal to $\frac{O\,d + O\,d'}{2}$. The minor axis has a length $N\,O$ equal to $\frac{O\,d' - O\,d.}{2}$ The ellipse can then be drawn by the paper trammel in the usual way.

72. Properties of the Central Ellipse and Ellipse of Inertia.*—The following properties of the central ellipse and ellipse of inertia are important.

1. If the reduced moments of a system of parallel forces about an axis X X are obtained, and these reduced moments are supposed to act as parallel forces at the original points of application; then the centre of reduced moments, if joined to any point O on X X, gives the direction of the diameter conjugate to X X of that ellipse of inertia which has O as centre.

* In order to avoid somewhat lengthy mathematical proofs, it has been thought best simply to state the most important properties of the Central Ellipse and Ellipse of Inertia. A much fuller treatment of this branch of the subject will be found in 'Elemente der Graphischen Statik,' Bauschinger, X. Abschnitt, and in 'Die Graphische Statik,' Culmann, II. Abschnitt, Kapitel 8.

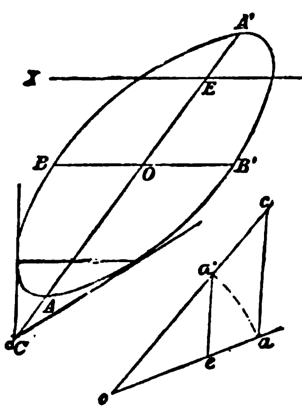
2. If the centre o of any curve of inertia of a system of parallel forces is joined to the centre O of the central ellipse, the diameters of the ellipse of inertia and of the central ellipse conjugate to the line o O are parallel and equal. The lengths of the other diameters (those which lie on o O) bear the following relation to each other. If A is the semi-diameter of the central curve, a that of the ellipse of inertia, and d the distance between their centres—

$$a^2 = A^2 + d^2$$

3. If any line X X is assumed as the direction of a diameter of the central ellipse of a system of parallel forces, and the latter are divided into two groups whose reduced moments about X X are determined, then if the centres of the reduced moments (§ 68) are separately determined for each group, the line joining these centres is parallel to that diameter of the central ellipse which is conjugate to X X.

4. The centre of the reduced moments about an axis X X of a system of parallel forces is in the central ellipse the *pole* of a line drawn parallel to X X at the same distance from the centre of forces as X X, but on the opposite side. (In any conic section, an external point is termed the "pole" of the chord of contact of the tangents drawn from it. An internal point is the "pole" of the locus of the intersection of pairs of tangents drawn from the extremities of all chords drawn through it.) The centre of reduced moments is therefore a point on that diameter of the central ellipse (produced if necessary) which is conjugate to the direction of the axis X X about which the moments are taken. Thus, if A B A′ B′ (Fig. 93) is the central ellipse of a system of parallel forces; then, C the centre of reduced moments of these forces about an axis X X, lies on the diameter A′ A produced, this diameter being conjugate to B B′, which is parallel to X X. Moreover, if O A = a; O E = d; and O C = x;—

FIG. 93.



$$x = \frac{a^2}{d},$$

from which the position of C can be determined if the central ellipse of the system, or merely its two conjugate diameters A′ A, B′ B are known, or have been obtained. The construction for determining x is shown.

5. If a system of forces can be split up into pairs such that the lines $a_1 a_2, b_1 b_2 \ldots$ joining the points of application of the forces forming each pair are all parallel, and if the centres of all pairs lie on a straight line X X, then X X is conjugate to the direction of $a_1 a_2, b_1 b_2, \ldots$ in the central curve of the system, and in all curves of inertia whose centres lie on X X. Similarly, if the forces forming a system can be split up into pairs of groups, A_1 A_2 being the centres of one pair of groups, B_1 B_2 the

centres of another pair, and so on; and if the centres of all pairs of groups lie on a straight line X X, then X X is conjugate to the directions $A_1 A_2$, $B_1 B_2$..... in the central curve of the system, and in all curves of inertia whose centres lie on X X. In either case the line X X obviously passes through the centre of the whole system. If the directions of a pair of conjugate diameters are at right angles to each other, these directions are those of the principal axes of the central curve.

6. If $x_1 y_1$, $x_2 y_2$, ... are the respective co-ordinates of the points of application a_1, a_2 .... of a system of parallel forces referred to two axes X X, Y Y, and if $\Sigma (x y) = o$, then X X and Y Y are conjugate diameters of the central curve of the system.

73. Moment of Inertia of a System of Forces determined by means of the Central Ellipses of its Groups.—In Fig. 94, E_1 is the central ellipse of a group of forces $p'_1 + p'_2 + . = \Sigma p'$, and E_2 the central ellipse of a group $p_1'' + p_2'' + . = \Sigma p''$, the two groups being portions of a system. It is required to determine the moment of inertia of the system about an axis $X_1 X_1$. The centre of the system O is supposed to be known, or to have been already obtained, as also Y Y the direction of the diameter conjugate to X X of the central ellipse of the whole system: X X being parallel to $X_1 X_1$. Dealing first with the group p'_1, p'_2..., draw $B_1 B_1'$ the diameter of its central ellipse E_1 parallel to X X, and determine $A_1 A_1'$ the diameter conjugate to $B_1 B_1'$. Produce $A_1' A_1$ to cut $X_1 X_1$ in C_1. Then the ellipse of inertia of the group p_1', p_2',.... will (§ 72. 2) have one diameter parallel and equal to $B_1 B_1'$, while the length of the other which lies on $A_1' C_1$ is the hypothenuse of a right-angled triangle whose sides are equal to $C_1 O_1$ and $A_1 O_1$. Draw $O_1 a_1$ equal and at right angles to $A_1 O_1$: make $C_1 D_1$ equal to $C_1 a_1$. Then D_1 is the extremity of the diameter conjugate to $X_1 X_1$ of that ellipse of inertia of the forces p_1', p_2'... which has C_1 as its centre, and the perpendicular from D_1 on $X_1 X_1$ is (§ 71) the radius of gyration (k_1) of the group p_1', p_2'... about the axis $X_1 X_1$. Hence (§ 70) $k_1^2 . \Sigma p'$ is the moment of inertia of this group about X X. Repeating the above construction for the central

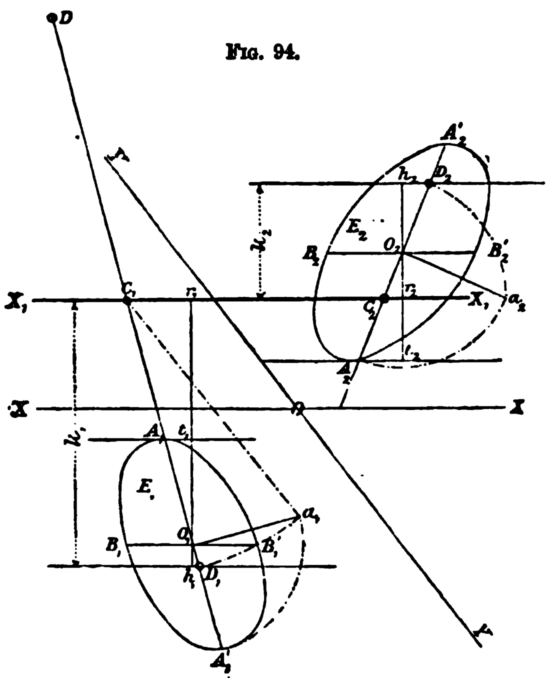


FIG. 94.

ellipse E_2 of the group of forces p_1'', p_2'', ..., with similar lettering $k_2^2 . \Sigma p''$ is the moment of inertia of this second group about $X_1 X_1$. The moment of inertia of the *whole system* about $X_1 X_1$ is therefore $k_1^2 . \Sigma p' + k_2^2 . \Sigma p''$; or, if there are more than two groups,
$$k_1^2 . \Sigma p' + k_2^2 . \Sigma p'' + k_3^2 . \Sigma p''' + \ldots .$$
Having determined the points D_1, D_2, D_3 ... for all the groups of a system it is only necessary to suppose the sums of the forces $\Sigma p'$, $\Sigma p''$... to act as parallel forces respectively at D_1, D_2,...($\Sigma p'$ at D_1; $\Sigma p''$ at D_2 and so on), and then to treat them precisely as single forces by the method of § 69. Since in this construction two funicular polygons will have to be drawn between lines passing through D_1, D_2, ... parallel to $X_1 X_1$, it is not absolutely necessary to find the positions of D_1, D_2... Thus, if a tangent to the ellipse E_1 is drawn parallel to $X_1 X_1$ and a perpendicular is let fall on this tangent from O_1 cutting the tangent in t_1 and the axis $X_1 X_1$ in r_1: then h_1, determined by making $\overline{h_1 r_1} = \overline{O_1 r_1}^2 + \overline{O_1 t_1}^2$, is the point through which to draw the parallel to $X_1 X_1$ and $h_1 r_1 = k_1$, the radius of gyration of the group p_1', p_2'... about $X_1 X_1$.

Instead of being supposed to act at D_1 the sum of forces $\Sigma p'$ may be taken to act at D, the other extremity of the diameter conjugate to the direction of $X_1 X_1$ of the ellipse of inertia which has O_1 as centre.

74. Ellipse of Inertia and Central Ellipse of a System of Five Parallel Forces.—A_1, A_2..A_5 (Fig. 94, Pl. IX.) are the points of application of five parallel forces $p_1 = 40$ lb.; $p_2 = 60$ lb.; $p_3 = 35$ lb.; $p_4 = 80$ lb.; $p_5 = 50$ lb. of similar sense; it is required to draw that ellipse of inertia of the system which has a given point Q as centre.

Draw any axis X X through Q; then (§ 72) the diameter of the ellipse of inertia conjugate to X X will pass through the centre of reduced moments of the forces about X X. Set off the forces successively on a load line 0 5, which can conveniently be taken on X X. Assuming the forces to act parallel to X X, draw the funicular polygon I...V with respect to the pole O. The sides of this polygon severally cut X X in 0′, 1′, ... 5′, thus determining 0′ 1′, 1′ 2′...the reduced moments of p_1, p_2, ... Suppose those reduced moments to act as forces parallel to X X and draw the funicular polygon I_1 ... V_1 with respect to the pole O′, which should be taken at a distance from X X equal to 0 5; *i e*, Σp. The extreme sides of this funicular polygon intersect in m_2. Now suppose the reduced moments to act as forces at right angles to X X, and draw the funicular polygon I_1'...V_1' with respect to the same pole O′. The sides of this polygon must of course be drawn at right angles to the vectors O′*o*′, O′ 1′... The first and last sides of this polygon intersect in m_1, then lines from m_2 and m_1 respectively, parallel and perpendicular to X X intersect (§ 69) in M the centre of reduced moments about X X. Thus M Q is the direction of the diameter of the ellipse of inertia conjugate to X X.

To find the length of this diameter it is necessary (§ 72) to obtain the radius of gyration (k) of the system about X X. By § 70, $k = \sqrt{ef \cdot OH}$; where e and f are the intersections of the first and last sides of the funicular polygon $I_1 \ldots V_1$ with X X, and O H is the polar distance in the original polygon of forces. A line parallel to and at a perpendicular distance k from X X therefore (§ 71) cuts Y Y; i.e. M Q produced, in the extremity of the required diameter of the ellipse of inertia.

In order to find the length of the other diameter it is necessary to obtain k_1 the radius of gyration about Y Y. The construction, which is precisely similar to that described above, is carried out by means of the two funicular polygons $I_2 \ldots V_2$ and $I_2' \ldots V_2'$, and the last sides of the latter cut Y Y in e_1, f_1. Then $k_1 = \sqrt{e_1 f_1 \cdot O_2 H_2}$ where $O_2 H_2$, the polar distance of the new polygon of forces, has been taken equal to O H. A line parallel to and at a perpendicular distance k_1 from Y Y cuts X X in the extremity of the other diameter, and the ellipse of inertia can now be drawn (§ 71).

The centre C of the system is determined by means of the two funicular polygons I . . . V and I′ . . . V′ (§ 67). Joining M the centre of reduced moments to C, the direction of the diameter of the central ellipse conjugate to a line through C parallel to X X is obtained. The radius of gyration about the axis through M and C, obtained precisely as above, determines the length of one semi-diameter of the central ellipse. The radius of gyration K about the conjugate diameter through C parallel to X X can be deduced from the already obtained radius of gyration (k) about X X, since $k^2 = K^2 + d^2$, where d is the perpendicular distance from C to X X.

To avoid confusion of lines the construction for the central ellipse is not carried out in Fig. 94, Pl. IX. This figure appears complicated, but its construction will be found extremely simple and it affords an excellent exercise in dealing with funicular polygons.

CHAPTER X.

MOMENT OF RESISTANCE.—CENTRAL ELLIPSE AND KERN OF A SECTION.

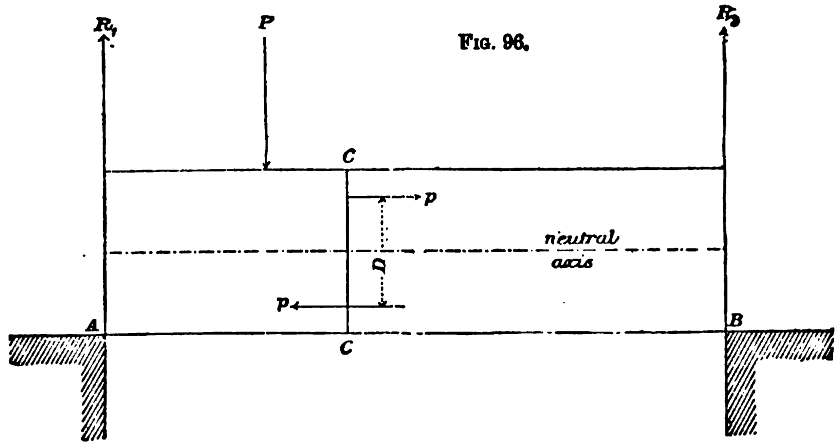
75. **Bending Stress.**—A beam defined as in § 41 is subject to simple bending stress only, when the loading can be reduced to a couple acting in the plane of symmetry containing the axis. If all, or any of the external loads act out of this plane, a stress due to twisting arises. If they are not normal to the axis, the beam has to

sustain *direct* in addition to *bending stress* (§ 48). If the loading can be reduced to a couple and a vertical force, then (§ 41) the beam is subject to bending and shearing stress. This is the case of a horizontal beam supported at both ends, and sustaining vertical loads acting in its plane of symmetry.

Considering the section C C of a beam A B (Fig. 96) supported at A and B, and sustaining a load P. The action of P gives rise at C C to a couple whose moment M ($= R_2 .$ B C, the *bending moment* at C C) tends to produce rotation contrary to the direction of the hands of a watch, and also to a vertical tangential force F ($= R_2$) acting in the plane of the section C C—the shearing force at C C—tending to cause the right segment C B of the beam to move upwards relatively to the left segment.

It has been shown in Chapters V. and VI. how the bending moment and shearing forces at any section of beams variously loaded can be obtained. It remains to deal

FIG. 96.



with the resisting forces, or stresses, so as to be able to ascertain whether a beam is sufficiently strong at a given section.

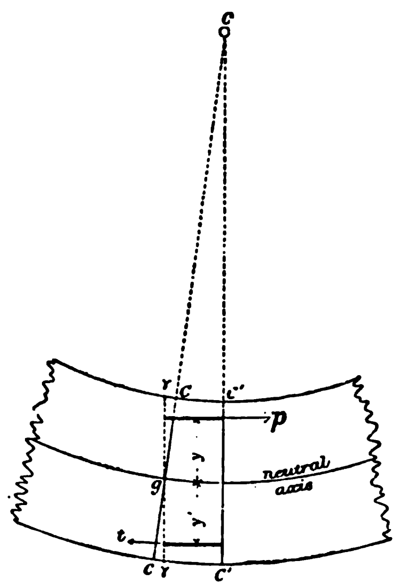
In the case of Fig. 96, for example, the bending moment M must be resisted by a stress couple whose moment is equal and opposite: this couple is the resultant of the molecular stresses in all the fibres cut by the section plane, and its moment is termed the *Moment of Resistance* of the section.

Again, the shearing force F must be met by an equal and opposite shearing stress, or molecular resistance of the section to shearing.

If the beam is bent under the loading, as shown in Fig. 97, its lower surface is evidently extended and its upper surface compressed. Between the upper and lower surface there is a layer of fibres whose lengths are unchanged, and which is termed the *Neutral Surface.* This layer, dividing fibres in tension from those in pressure, cuts the plane of symmetry of the beam in the *Neutral Axis of the Beam*, and the plane of the section C C in the *Neutral Axis of the Section.*

In Fig. 97 C C, C′ C′ are supposed to be two sections originally parallel and indefinitely near to each other. Through g, the point where C C cuts the neutral axis of the beam, $\gamma\gamma$ is drawn parallel to C′ C′. Then evidently, if the curve taken by the neutral axis is considered to be a circle with O as centre, the contraction, or loss of length in the upper fibres is proportional to their distances from the neutral axis, and similarly for the elongation, or gain of length in the lower fibres. The actual bending is supposed to be very small, so that the straining forces remain sensibly parallel, and the resultant of the pressures above and the tensions below the neutral axis are equal and opposite. These resultants form a couple, whose moment p . D (Fig. 96) is the moment of resistance of the section.

Fig. 97.



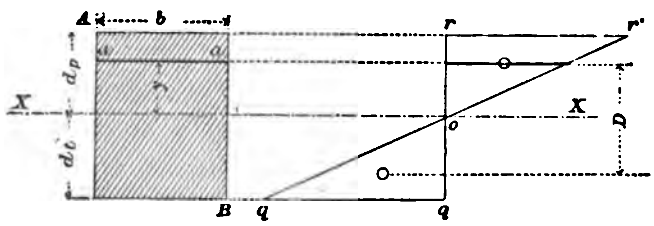
But, provided that the limit of elasticity is not exceeded, the elongations and contractions are proportional to the tensions and pressures producing them. If, therefore, t and p are the tension and pressure respectively in fibres distant y' and y from the neutral axis—

$$t \propto y' \text{ and } p \propto y.$$

Hence if A B (Fig. 98) represents the section C C (Figs. 96 and 97), and X X its neutral axis, the intensity of stress in a layer of fibres $a\,a$ distant y from X X is $c \,.\, y$, where c is a constant.

If a is the area of the layer, its breadth being supposed very small, the total stress (s) in the layer is $c \,.\, a \,.\, y$; the moment of this stress about X X is $c \,.\, a \,.\, y^2$, and the total moment about X X of the stresses in all the fibres cut by the section plane is $\Sigma\,(c \,.\, a \,.\, y^2)$, or $c \,.\, \Sigma\,(a\,y^2)$. If $a = 1$ and $y = 1$, then $s = c$: hence the constant c is the

Fig. 98.



stress per unit of area at a unit distance from the neutral axis. (Since no other horizontal forces come into play, the sum of the tensions is equal to the sum of the pressures, or $s . \Sigma\,(a\,y) = 0$: hence the neutral axis in the case of simple bending passes through the centre of gravity of the section.)

The total moment of resistance of the section is $s \,.\, \Sigma\,(a\,y^2)$. If d_p and d_t are the distances of the extreme fibres in pressure and tension respectively, and f_p, f_t the safe working stress per unit of area of the material in pressure and tension—

$$\frac{s}{1} = \frac{f_p}{d_p}; \text{ and } \frac{s}{1} = \frac{f_t}{d_t}.$$

Hence,

$$s = \frac{f_p}{d_p}, \text{ or } \frac{f_t}{d_t},$$

and the moment of resistance of the section is

$$\frac{f_p}{d_p} \Sigma (a y^2), \text{ or } \frac{f_t}{d_t} \Sigma (a y^2).$$

Thus, in order that the section may have sufficient resistance, M, the bending moment at the section, must be equal to or less than the lesser of the two values $\frac{f_p}{d_p} . \Sigma (a y^2)$ or $\frac{f_t}{d_t} . \Sigma (a y^2)$. If the material is equally strong to resist pressure and tension, as in the case of wrought iron and steel, $f_p = f_t$. If the section is symmetrical about its neutral axis, $d_p = d_t$.

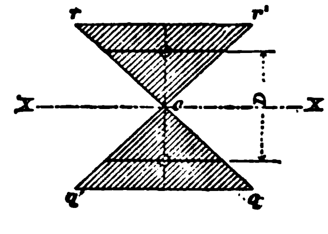
76. **Resistance Area.** — If the position of the neutral axis of a section is known, and if the stresses are proportional to the distances from this axis, then evidently a figure can be drawn which represents in magnitude and distribution the stresses over the section at the moment when the extreme fibres on one side are strained to the extent of their safe load. Such a figure is termed the *Safe Resistance Area* of the section.

In the case of the rectangular section (Fig. 98), since the breadth is everywhere the same, the stress in any layer of fibres *a a* is simply proportional to the distance *y* of that layer from X X. Draw *r q* equal to the depth of the section and perpendicular to the neutral axis. Then if f_p is the safe load per unit of area of the material in pressure and tension, and *b* the breadth of the section, $f_p . b$ is the stress in the upper layer of fibres. Set off *r r'* and *q q'* at right angles to *r q* and equal to $f_p . b$, read off from any convenient scale which may be termed the scale of resistances: join *r' q'*. Then the whole area *r r' q' q* represents the stress over the section A B in magnitude and distribution. The area of the triangle *o r r'* represents the sum of the pressures, and the latter have a resultant passing through its centre of gravity. Similarly *o q q'* represents the sum of the tensions. The ordinate of the figure *r r' q' q* opposite any horizontal layer *a a* gives the stress in that layer.

If now the stress f_p is supposed to act uniformly over the whole area *r r' q' q*, as a pressure over *o r r'* and a tension over *o q q'* : then the moment about X X of resistance of the whole figure *r r' q' q* with its varying stress is the same as that of the section A B with its varying stress. In the present case, as the section has a horizontal and vertical axis of symmetry, the pressure and tension areas are similar as well as equal.

They will of course be dissimilar in the case of sections which are not thus symmetrical, but they must always be equal, since the resultant pressures and tensions form a couple. In Fig. 98 the centres of gravities of the two triangles give the distances* of the points of application of the resultant pressures and tensions from X X; and if D is the *vertical* distance between these centres of gravity, and A the area of one of the equal triangles, the moment of resistance is D . A.

FIG. 99.



If $r r'$ is taken as equal to b, or if f_p is taken as unity on the scale of resistance, then the resistance area becomes an *equivalent area* to the section; i. e. the area which, with the same material and uniform in place of a varying stress, would have the same moment of resistance. The moment of resistance in this case would be $D . A . f_p$. It may sometimes be convenient to follow this plan if it is desired to form a comparison of the resistance areas of different sections of the same material.

The beam tends to fail by *pressure* or *tension*, according as $\frac{f_p}{f_t}$ is *less* or *greater* than $\frac{d_p}{d_t}$. If, therefore, the resistance area is drawn (1) for safe stress in *pressure*; (2) for safe stress in *tension*: then that area for which *neither* of the limiting stresses exceeds the safe working load of the material, will be the one corresponding to the actual load which the section can safely bear, or the *safe resistance area* of the section.

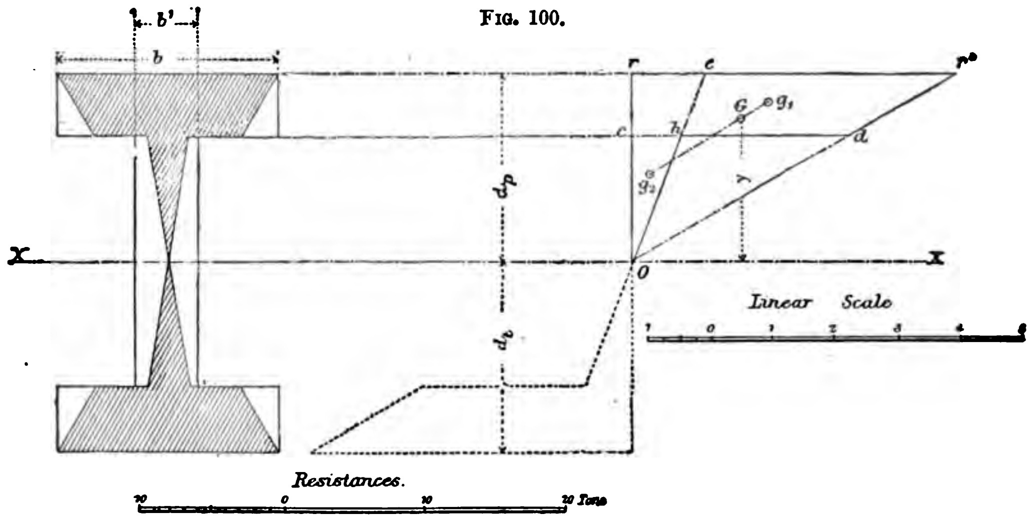
77. Examples of Resistance Areas. — To apply the preceding section practically, suppose that the moment of resistance of the **I** section given in Fig. 100 is to be determined by means of a resistance area. The latter to be drawn for a safe working stress in pressure and tension of 5 tons per square inch. Draw $r o$ as before at right angles to X X, and $r r'$ at right angles to $r o$, making $r r' = b \times 5 = 22\frac{1}{2}$ tons. Join $r' o$, and draw a horizontal line through the lower edge of the upper flange cutting $r o$ in c and $r' o$ in d; then $r c d r'$ is the resistance area of the upper flange. Make $r e$ equal to $b' \times 5$ (= 5 tons), and join $e o$, cutting $d c$ in h. Then $\frac{c h}{r e} = \frac{c o}{r o}$ and $c h$ represents (on the same scale as $r r'$) the resistance of the top layer of fibres of the web of the section. Thus the figure $o r r' d h$ is the resistance area of the upper half of the **I** section. The resistance area of the lower half is precisely similar, and

* As drawn, the tension area is shown to the left, and the pressure area to the right, of a vertical $r q$; this plan possesses some advantages, but has the effect of equally displacing the centres of gravity of these areas left and right of the vertical. As long as the distances of these centres of gravity from $r q$ are equal, therefore, the stresses have no moment about $r q$. The resistance area might of course be drawn as shown in Fig. 99, where it is symmetrical about a vertical line.

need not be drawn. Find g_1 and g_2 (§ 65), the centres of gravity of the trapezium $r\,c\,d\,r'$, and the triangle $c\,o\,h$ * respectively, and divide $g_1\,g_2$ in G such that

$$\frac{g_1 G}{g_2 G} = \frac{\text{area of } c\,o\,h}{\text{area of } r\,c\,d\,r'};$$

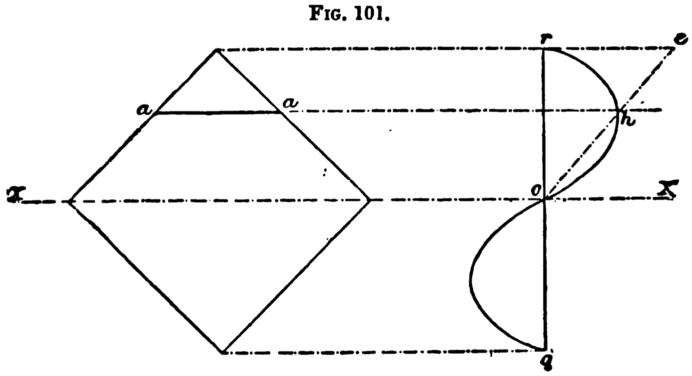
FIG. 100.



then G is the centre of gravity of the whole resistance area; and if y is the vertical distance of G from X X, the moment of resistance (M) of the **I** section is—

$$2\,y \,.\, \text{area } o\,r\,r'\,d\,h = 2\,y \,.\, \tfrac{1}{2}\,(r\,r' \,.\, r\,o - h\,d \,.\, c\,o).$$

FIG. 101.



It is only necessary now to read off $r\,r'$ and $h\,d$ from the scale of resistances, and y, $r\,o$, and $c\,o$ from the linear scale, and we obtain—

$$M = 97 \cdot 87 \text{ inch tons},$$

which must not be *less* than the bending moment at the section under consideration expressed in the same units.

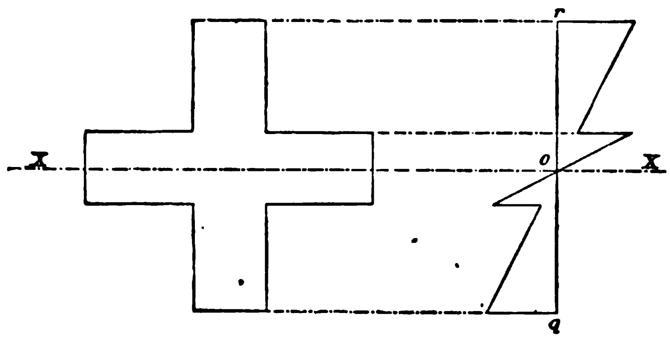
If an equivalent area is to be drawn, $r\,r'$ must be made equal to b and $r\,e$ to b'.

* The centre of gravity of $o\,r\,r'\,d\,h$ may also be found by considering it to be the *difference* of the triangles $o\,r\,r'$ and $o\,h\,d$.

Or the equivalent area may be drawn symmetrical about the vertical axis of the I section, and superposed on the section itself, as shown in Fig. 100. The construction needs no explanation.

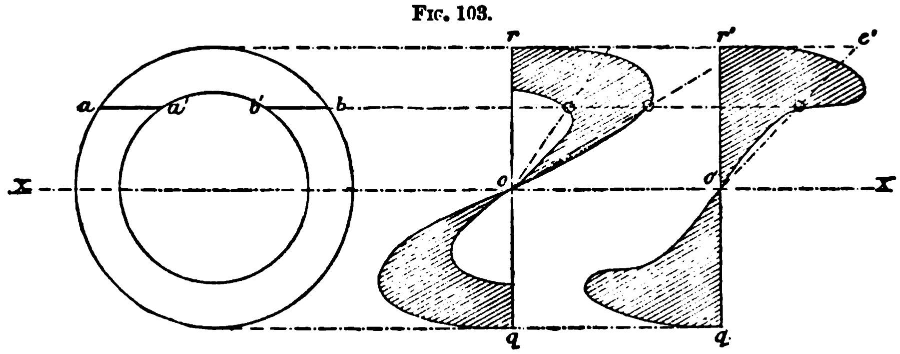
In Figs. 101, 102, 103, and 104, equivalent areas are shown for sections of given forms. It will be noticed that wherever the outline of the section is perpendicular to

FIG. 102.



X X, the corresponding outline in the resistance area is a line passing through *o*. When the outline is parallel to X X in the section, it is parallel to X X in the resistance area. Any other straight line in the section gives an arc of a parabola in the resistance area. In other cases the outline of the latter is a curve, any number of points on which can be obtained by dealing with separate horizontal layers of fibres in the section. Thus, taking the layer *a a* (Fig. 101), make *r e* equal to *a a* : join *e o*,

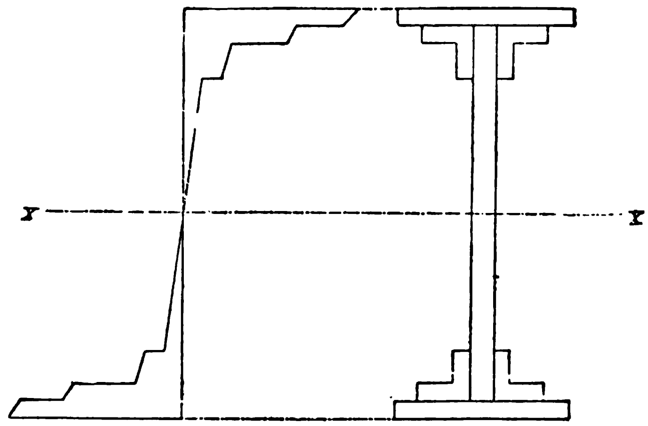
FIG. 103.



cutting *a a* produced in *h*. Then *h* is a point on the curve outline, and any number of points can similarly be obtained. In the case of the hollow circular section (Fig. 103), points on the outline of the equivalent area can be obtained by treating the section as the difference of the two circles, setting *a b* and *a′ b′* from *r* along *r r′*, and drawing lines to *o* cutting *a b* produced. Or $a a' + b' b (= 2 \,.\, a a')$ can be set off from *r′* to *e′*, and *e′ o′* joined. A different outline is of course obtained, but the same equivalent area. Both outlines are shown in Fig. 103.

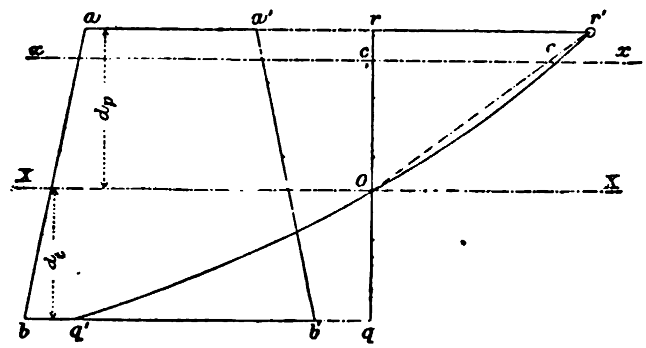
If the section has only a vertical axis of symmetry, as in the case of the trapezium (Fig. 105), it will be necessary first to determine its centre of gravity (§ 65), and then to draw the neutral axis X X. Two resistance areas can now be drawn; one for safe stress in tension, the other for safe stress in pressure (§ 76). The lower half, or tension area, of the former will be obtained precisely as before. In the present instance an

Fig. 104.



equivalent area has been drawn, and $q\,q'$ is therefore made equal to $b\,b'$. Points on the curve $o\,q$ are then obtained as before. For the upper half, or pressure area, draw $x\,x$ parallel to X X; and at a distance from it, equal to d_t, the depth of the extreme extended layer of fibres from the neutral axis X X. Then make the reduction of

Fig. 105.

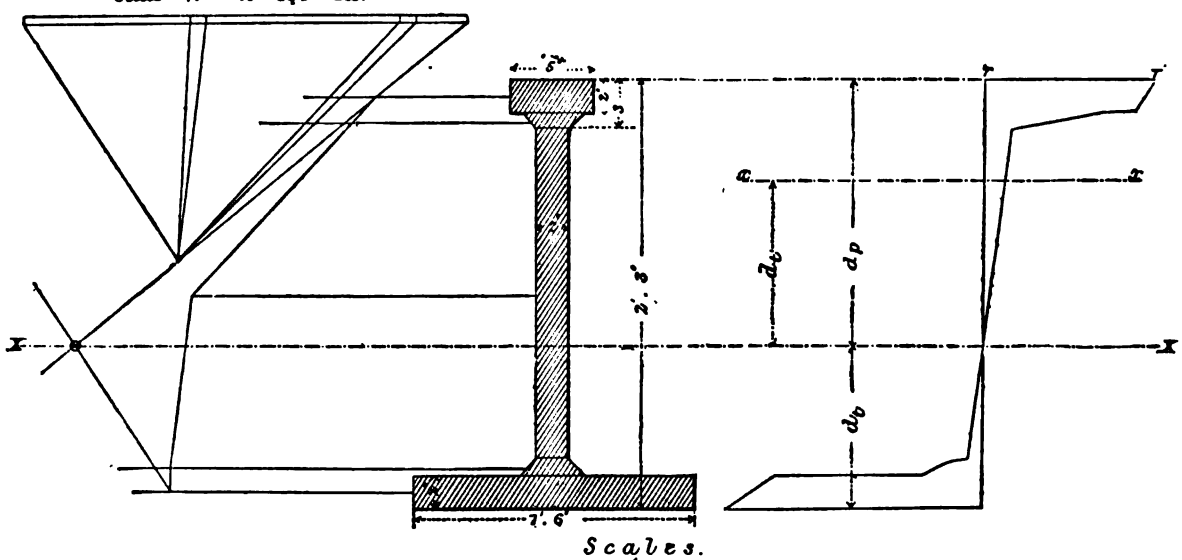


resistances on the line $x\,x$. Thus, for the upper layer of fibres $a\,a'$, set off $c\,e$ on $x\,x$ equal to $a\,a'$: join $o\,e$, and produce $o\,e$ to cut $a\,a'$ produced in r'. Then $r\,r'$ represents the pressure in the upper layer of fibres $a\,a'$. If this pressure is found to be greater than the assumed safe stress of the material in pressure, the section tends to fail by pressure. The construction of an equivalent area for safe stress in pressure is precisely similar. A line must be drawn parallel to X X, and a distance from it equal to d_p: on this line, the reductions of resistances for the lower half of the section must be made. If the areas have been drawn as resistance areas, and in one of them neither of the limiting stresses is found to exceed the safe working load of the material, this will be the resistance area corresponding to the safe load of the section.

It will be evident that by reversing the process above described a section can be drawn to correspond to any given or assigned distribution of stress.

In Fig. 106 a resistance area for a safe stress in tension of 1·5 ton per square inch has been drawn, in the case of the section of a cast-iron beam. The centre of gravity of the section is first found by means of a funicular polygon. The section is cut up into five parts, as shown, and the number of square inches in each part, obtained arithmetically, is read from any convenient scale, and set off along a load line in the usual way. The reductions for the pressure area are made to a line $x\,x$ parallel

FIG. 106.



to X X, and a distance d_t from it. The pressure in the upper layer of fibres at the moment when the lower layer is subject to a tension of 1·5 ton per square inch is represented by $r\,r'$, and the stress per unit of area in the upper is $\frac{r\,r'}{5} = \frac{12}{5} = 2 \cdot 4$ tons, $r\,r'$ being read off from the resistance scale.

78. **Moment of Inertia of a Section.**—An area considered as made up of heavy elements can be said to have a moment of inertia in the same sense as a centre of gravity. If $a_1, a_2 \ldots$ are the areas of indefinitely small elements, and $y_1, y_2 \ldots$ their respective perpendicular distances from an axis X X, then $\Sigma\,(a\,y^2)\ (= \mathrm{I})$ is the moment of inertia of the whole area about X X. If $\Sigma\,a$ (the whole area of the figure) $= \mathrm{A}$, then $\frac{\Sigma\,(a\,y^2)}{\mathrm{A}} = k^2$, where k is the radius of gyration of the area about X X. Thus k is the distance from X X of a point such that the whole area, if considered there concentrated, would have the same moment of inertia as the actual distributed area.

If a figure is cut up into several portions, the moment of inertia of the whole figure about any axis is equal to the sum of the moments of inertia of the several portions about the same axis. If a section is conceived as the difference of two figures, the moment of inertia of the section about any axis is equal to the difference of the moments of inertia of the two figures about the same axis. If I is the moment of inertia of an area (A) about an axis through its centre of gravity, and I′ its moment of inertia about a parallel axis at a distance d from the first, then—

$$I' = I + A \cdot d^2.$$

Thus, if the moment of inertia of a few simple figures is known, that of a more complex figure can be obtained, and from the moment of inertia about one axis that about any parallel axis can be deduced.

79. Moment of Inertia of a Section determined by the Funicular Polygon.— If a section is cut up into strips parallel to any axis, and the areas of the strips are supposed to act as parallel forces at their respective centres of gravity, then the moment of inertia of the system obtained as in § 69 will be approximately the moment of inertia of the area, the approximation being closer the greater the number of strips thus dealt with.

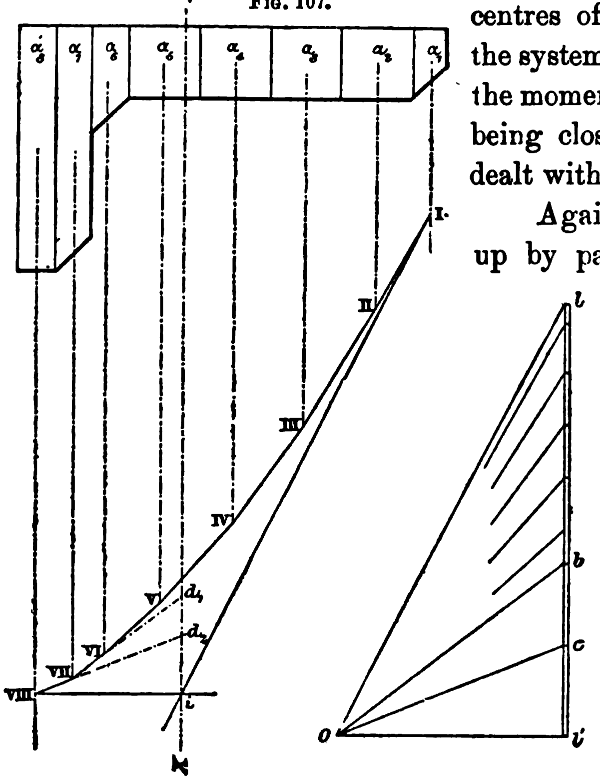


Fig. 107.

Again, suppose the section (Fig. 107) to be cut up by parallel lines into a number of strips, as shown. Set off the areas of these strips $a_1, a_2, a_3 \ldots$ along a load line $l\,l'$, and supposing $a_1, a_2, a_3 \ldots$ to act as parallel forces at the centres of gravity of the respective strips, draw the funicular polygon I II III... with a polar distance equal to $\frac{\Sigma a}{2}\left(=\frac{l\,l'}{2}\right)$. The first and last sides of the funicular polygon intersect on X X, the neutral axis of the section. Produce any pair of adjacent sides VII VI, VIII VII of the funicular polygon to cut X X in d_1, d_2. Let $y_1, y_2 \ldots$ be the distances of the centres of gravity of the strips $a_1, a_2, \ldots$ from X X respectively. Then since the triangles $b\,c$ O, $d_1\,d_2$ VII are similar—

$$\frac{d_1\,d_2}{y_7} = \frac{b\,c}{o\,l'} = \frac{a_7}{\frac{\Sigma a}{2}}.$$

Hence—

$$d_1 d_2 \cdot \frac{a_7}{2} = \frac{d_7 \cdot y_7^2}{\Sigma a}.$$

But $d_1 d_2 \cdot \frac{y_7}{2}$ is the area of the triangle VII $d_1 d_2$, and the whole figure contained by the sides of the funicular polygon is made up of all the triangles corresponding to VII $d_1 d_2$. Hence if A_1 is the area of this figure—

$$A_1 = \frac{\Sigma (a y^2)}{\Sigma a},$$

or

$$I = A_1 \cdot A,$$

when A is the area of the section, and I its moment of inertia about X X.*

80. **Moment of Inertia of a Section obtained from the Resistance Area.**— It was shown (§ 75) that the moment of resistance (M) of a section about its neutral axis is $\frac{f_p}{d_p} \Sigma (a y^2)$ or $\frac{f_t}{d} \Sigma (a y^2)$; i.e. $\frac{f_p}{d_p} \cdot I$ or $\frac{f_t}{d_t} \cdot I$, where I is the moment of inertia of the section about its neutral axis. By § 76, $M = D \cdot A \cdot f_p$, where A is the equivalent tension or pressure area, and D the distance apart of their centres of gravity.

Hence—

$$D . A . f_p = \frac{f_p}{d_p} \cdot I,$$

and

$$I = d_p \cdot D \cdot A.$$

Thus, the moment of inertia of any section can be determined from its equivalent area. In the case of the rectangular section (Fig. 99), for instance—

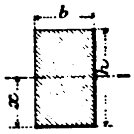
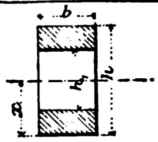
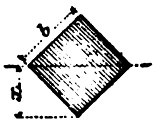
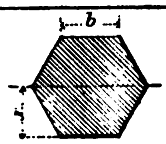
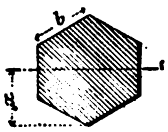
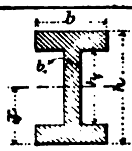
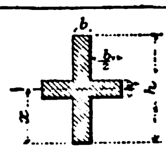
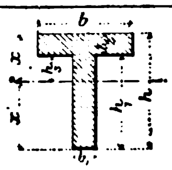
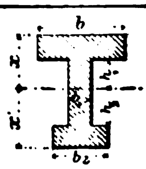
$$I = \frac{h}{2} \cdot \tfrac{2}{3} h \cdot b \cdot \frac{h}{4} = \frac{b h^3}{12},$$

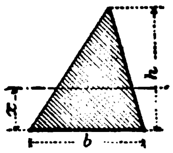
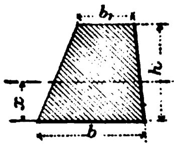
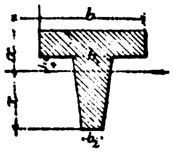
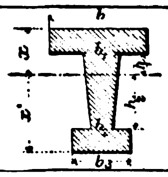
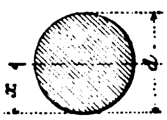
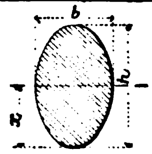
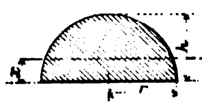
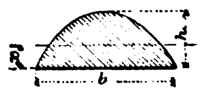
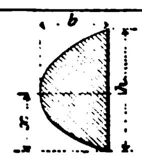
where b is the breadth and h the height of the section: if $b = h$ the section becomes a square, and

$$I = \frac{h^4}{12}.$$

(The moment of inertia of figures bounded by regular curves is most readily obtained by means of the integral calculus.) The following table gives the values of I for sections of various forms:—

* This construction must be considered as merely approximate. An error is introduced in supposing the reduced moments to act as parallel forces at the centres of gravity of the laminas. These moments should be taken to act at the centres of moments about X X of the elementary areas which make up each lamina. For laminas at a distance from X X, and for laminas whose breadths are very small, these centres of moments and the centres of gravity nearly coincide. The approximation is more close therefore, the smaller the laminas are taken.

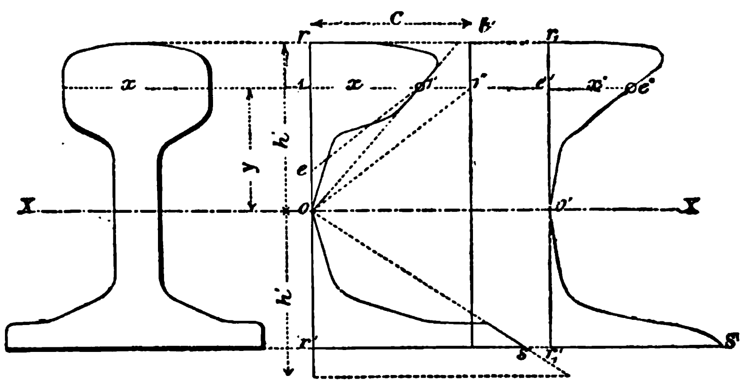
No.	Section.	Moment of Inertia	Area.	Distance x.
I.		$\frac{b\,h^3}{12}$	$b\,h$	$\frac{h}{2}$
II.		$\frac{b\,(h^3 - h_1^3)}{12}$	$b\,(h - h_1)$	$\frac{h}{2}$
III.		$\frac{b^4}{12}$	b^2	$\frac{b}{\sqrt{2}}$
IV.		$\frac{5\sqrt{3}}{16} \,.\, b^4 = 0{\cdot}5413\,b^4$	$\frac{3\sqrt{3}}{2} b^2 = 2{\cdot}598\,b^2$	$b\sqrt{\tfrac{3}{4}} = 0{\cdot}866\,b$
V.		$\frac{5\sqrt{3}}{16} \,.\, b^4$	$\frac{3\sqrt{3}}{2} b^2$	b
VI.		$\frac{b\,h^3 - (b - b_1)\,h_1^3}{12}$	$b\,h - (b - b_1)\,h_1$	$\frac{h}{2}$
VII.		$\frac{b\,h^3 + b_1\,h_1^3}{12}$	$b\,h + b_1\,h_1$	$\frac{h}{2}$
VIII.		$\frac{1}{3}[b\,(x^3 - h_2^3) + b_1\,(h_2^3 + x'^3)]$	$b_1\,h_1 + b\,h_2$	$\frac{b\,h_2 + b_1\,h_1\,(h + h_2)}{2\,[b\,h - (b - b_1)\,h_1]}$
IX.		$\frac{1}{3}[b\,(x^3 - h_1^3) +$ $b_1\,(h_1^3 + h_2^3) + b_2\,(x'^3 - h_2^3)]$	$h\,(x - h_1) + b_1\,(h_1 + h_2)$ $+ b_2\,(x' - h_2)$	Best found by construction.

No.	Section.	Moment of Inertia.	Area.	Distance x.
X.		$\frac{b h^3}{36}$	$\frac{b h}{2}$	$\frac{h}{3}$
XI.		$\frac{b^2 + 4 b b_1 + b_1^2}{36 (b + b_1)} \cdot h^3$	$\frac{b + b_1}{2} \cdot h$	$\frac{b + 2 b_1}{b + b_1} \cdot \frac{h}{3}$
XII.		$\frac{1}{3} \left[\frac{b_1 - b_2}{4 (h_1 + x')} (x'^4 - h_1^4) + b (x^3 - h_1^3) + b_2 (h_1^3 + x'^3) \right]$	$b (x - h_1) + \frac{b_1 + b_2}{2} (h_1 + x')$	Best found by construction.
XIII.		$\frac{1}{3} \left[\frac{b_1 - b_2}{4 (h_1 + h_2)} (h_2^4 - h_1^4) + b (x^3 - h_1^3) + b_2 (h_1^3 + h_2^3) + b_3 (x'^3 - h_2^3) \right]$	$b (x - h_1) + \frac{b_1 + b_2}{2} (h_1 + h_2) + b_3 (x' - h_2)$	Best found by construction.
XIV.		Circle. $\frac{\pi}{64} d^4 = 0 \cdot 0491\, d^4$	$\frac{\pi}{4} d^2$	$\frac{d}{2}$
XV.		Ellipse. $\frac{\pi}{64} b h^3$	$\frac{\pi b h}{4}$	$\frac{h}{2}$
XVI.		Semicircle. $0 \cdot 110\, r^4$	$\frac{\pi r^2}{2}$	$0 \cdot 4244\, r$
XVII.		Parabolic segment. $\frac{8}{175} b h^3 = 0 \cdot 0457\, b h^3$	$\frac{2}{3} b h$	$\frac{2}{5} h$
XVIII.		Parabolic segment. $\frac{1}{30} b h^3$	$\frac{2}{3} b h$	$\frac{h}{2}$

The following construction for determining the moment of inertia may sometimes be found convenient in dealing with figures of irregular outline. The case taken is the rail section (Fig. 107A), the moment of inertia I of which about the axis X X is to be determined.

Draw the axis $r\,o\,r'$ at right angles to X X′ and construct the curve $r\,1'\,0$ for the upper portion of the section precisely as an equivalent area: complete the curve for the lower portion, making the reductions to a layer of fibres at a distance h' from the

Fig. 107A.



axis X X equal to the distance of the top of the section from X X. In the figure the curves are drawn on the same side of $r\,r'$. Let x be any ordinate of the section parallel to X X; $1\,1'$ ($= x'$) the corresponding ordinate of the equivalent area; y the distance of x from X X. From r draw any length $r\,t$ ($=$ C) parallel to X X, and from t draw a line perpendicular to X X, cutting $1\,1'$ produced in $1''$. Join $1''\,0$, and from $1'$ draw a line parallel to $1''\,0$ cutting $r\,r'$ in e. From a new axis $r_1\,r_1'$ set off an ordinate $e'\,e''$ equal to $1\,e$, on $1\,1'$ produced. Let $e'\,e'' = x''$.

Now

$$\frac{x'}{y} = \frac{x}{h'} \text{ and } \frac{x''}{y} = \frac{x'}{C}.$$

Hence

$$x'' = \frac{x\,y^2}{h'\,C} \quad \ldots\ldots \quad (a).$$

Repeat the above construction for a sufficient number of ordinates of the curved outline $r\,1'\,o\,s$, and draw the new outline $r_1\,e''\,o'\,s'$ through the points obtained. Let A be the whole area contained by this second outline and the axis $r_1\,r_1'$.

By (a)

$$\Sigma(x'') = \frac{1}{h'\,C}\,\Sigma(x\,y^2);$$

but,

$$\Sigma(x\,y^2) = I \text{ and } \Sigma(x'') = A.$$

Hence

$$I = A \,.\, h' \,.\, C.$$

If, therefore, A is obtained by a planimeter, or by cutting the figure up into strips parallel to X X, and considering the strips as trapeziums, the moment of inertia of the rail section about X X is determined. The construction is very simple, and when carried out it remains to determine the area of a figure merely, and not an area and a centre of gravity as in the previous construction.

If X X is the neutral axis of the section, the moment of resistance is $\frac{f_p}{h'} . I = f_p . A . C.$

The figure A also gives the moment of inertia about X X of any portion of the section cut off by a line parallel to X X, for evidently the moment of inertia of the portion above x is equal to $h' . C \times$ area $r_1 e' e''$.

81. **Central Ellipse and Kern of a Section.**—In the sense in which an area is said to have a centre of gravity and a moment of inertia, it will also have a central ellipse conforming to the definition of § 71, and possessing all the properties of the central ellipse of a system of parallel forces stated in § 72.

The central ellipse for any section having been once drawn, the moment of inertia or radius of gyration of the section about *any axis whatever* lying in its plane can be at once obtained (§ 71). To draw the central ellipse of an area, it will in general be necessary to know the radius of gyration about two conjugate axes passing the centre of gravity of the area. If the area—the section of an **I** or **T** shaped beam, for example—has a line of symmetry, this line will be one of the principal axes of the central ellipse. In this case it will only be necessary to know the radius of gyration about the line of symmetry, and about a second line through the centre of gravity of the area, and at right angles to the first: these radii of gyration are the semi-major and semi-minor axes of the central ellipse. In other cases, such as a trapezium or triangle, the directions of two conjugate diameters can be obtained by inspection (§ 72–5). The radii of gyration about these diameters as axes give respectively the lengths of the diameters, provided that the lever arms (§ 68) are taken parallel to the direction of those diameters, whence the ellipse can be drawn (§ 71). If the lever arms are taken perpendicular to the axes, the radii of gyration give the perpendicular distances from the extremity of each diameter on the direction of the other.

Suppose that a system of equal parallel forces $p_1, p_2 \ldots$ act perpendicular to a section at points infinitely near to each other; in other words, suppose a uniformly distributed stress to act over the section, then the resultant stress passes through the centre of gravity (G) of the section; and if X X is any axis in the plane of the section $y_1, y_2 \ldots$, the distances from X X of the points of application of $p_1, p_2 \ldots$, and Y the distance of G from X X—

$$\Sigma(py) = Y . \Sigma p \ldots\ldots\ldots (a).$$

Suppose now that the original forces $p_1, p_2 \ldots$ are replaced by their moments

$p_1 y_1, p_2 y_2, \ldots$ about X X; i. e. suppose the section to be acted upon by a uniformly varying stress—a stress which varies directly as the distance from X X. The new system of forces will have a centre G′, and if Y′ is the distance of G′ from X X—

$$\Sigma (p y^2) = Y' . \Sigma (p y).$$

Substituting from (*a*)—

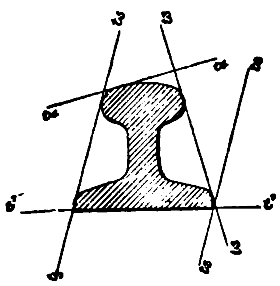
$$\Sigma (p y^2) = Y' . Y . \Sigma p.$$

The point Y′ at which the resultant of the uniformly varying stress acts may be termed the *Stress Centre*. This stress centre corresponds exactly to the centre of the reduced moments (§ 68) of a system of isolated forces.

If the axis about which the moment of inertia of a section is taken is supposed to turn about a fixed point P, the *locus* of the corresponding positions of the stress centre is a straight line, which in the central ellipse is the *polar* of a point P′; P′ and P being symmetrically situated with respect to the centre of gravity of the section.

Suppose now that the axis, about which the moment of inertia of a section is taken, moves round the section, always touching it, but never cutting it; taking, for example, the successive positions $t_1 t_1$, $t_2 t_2$, $t_3 t_3 \ldots$ (Fig. 108). Then the *locus* of the corresponding positions of the stress centre is a closed figure, termed the "*Kern*" of the section. It follows from the above that the kern of any rectilinear figure is itself a rectilinear figure having the same number of sides. The Central Ellipse and the Kern being geometrically related, either can be obtained from the other. Suppose that a plane figure is suspended vertically in a liquid so as to be always wholly immersed, but always touching the surface: then the Kern is the *locus* of the *Centre of Pressure* for all positions fulfilling the above conditions.

FIG. 108.



82. **Examples. Central Ellipse and Kern of Simple Figures.** — *Parallelogram.*—The lines X X, Y Y bisecting pairs of parallel sides (Fig. 109) are the directions of two conjugate axes of the central ellipse. The moments of inertia of the parallelogram about X X and Y Y—if the lever arms are taken parallel to Y Y, X X—are $\frac{b a^3}{12} . \sin \phi$ and $\frac{a b^3}{12} . \sin \phi$ respectively: the area of the figure is $a b . \sin \phi$, where A B′ = a and A B = b. Hence if k_1 and k_2 are the radii of gyration about X X and Y Y, and parallel to Y Y, X X respectively,

$$k_1^2 = \tfrac{1}{12} a^2; \text{ and } k_2^2 = \tfrac{1}{12} b^2.$$

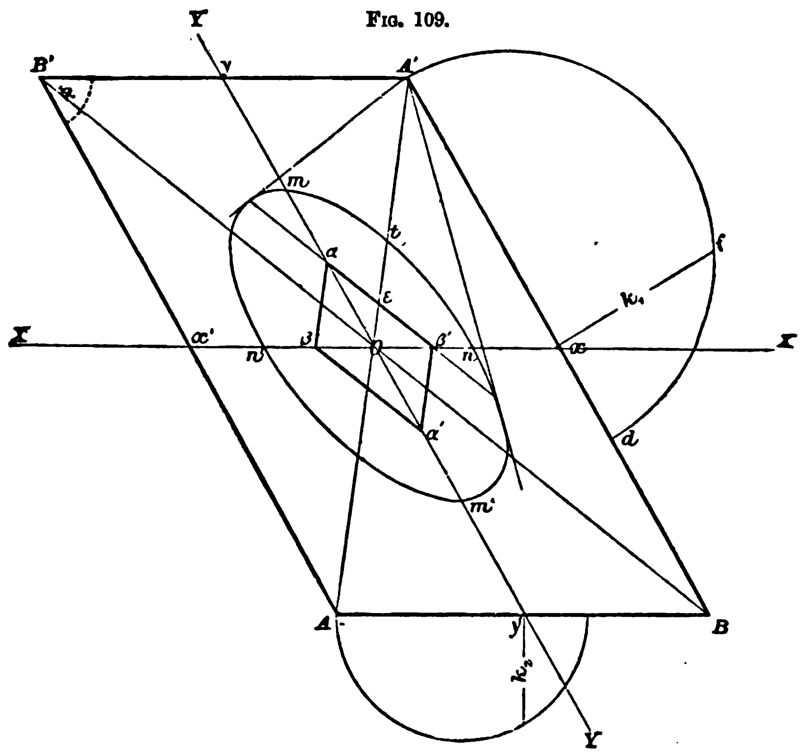
Thus

$$k_1 = \sqrt{\tfrac{1}{2} a . \tfrac{1}{6} a}, \text{ and } k_2 = \sqrt{\tfrac{1}{2} b . \tfrac{1}{6} b}.$$

Make $x\,d$ equal to $\frac{1}{3}$ A′ x and describe a semicircle on A′ d, cutting the perpendicular at x to A′ d in f: then $x f = k_1$. Make O m equal to k_1 and O n equal to k_2, determining k_2 in precisely the same way. The ellipse can then be drawn (§ 71).

The kern will be a four-sided rectilinear figure, for the stress centre describes a line while the axis turns about the angles of the parallelogram, and is a point when the

FIG. 109.



axis coincides with the sides of the parallelogram. When the axis turns about A the locus of the stress centre is the polar of A′. Draw tangents to the central ellipse from A′ then $\alpha\,\beta'$ the portion of the chord of contact intercepted by Y Y, X X is one side of the kern. The opposite side $\alpha'\,\beta$ is traced when A′ is the fixed point about which the axis turns: it is the polar, therefore, of A, and obviously from the symmetry of the figure O β' = O β and O α' = O α. Join $\alpha\,\beta$, $\beta'\,\alpha'$ and the kern is complete.

Again (§ 72–4)—

$$^{*}\,O\,\alpha = \frac{k_1^2}{O\,y} = \frac{\frac{1}{12}a^2}{\frac{1}{2}a} = \frac{1}{6}a.$$

Similarly—

$$O\,\beta = \tfrac{1}{6}b.$$

* It is easy to show that this is the case. By the preceding §, if K is the radius of gyration about an axis A B, Y and Y′ being the respective distance of the centre of gravity and the centre of stress of the section from A B, then—

$$K^2 . \Sigma p = Y . Y' \Sigma p; \text{ or } K^2 = Y . Y'.$$

But, if k_1 is the radius of gyration about an axis through O parallel to A B—

$$K^2 = k_1^2 + \overline{O\,y}^2.$$

Hence—

$$k_1^2 + \overline{O\,y}^2 = O\,y\,(O\,\alpha + O\,y)$$

and

$$k_1^2 = O\,\alpha\,.\,O\,y.$$

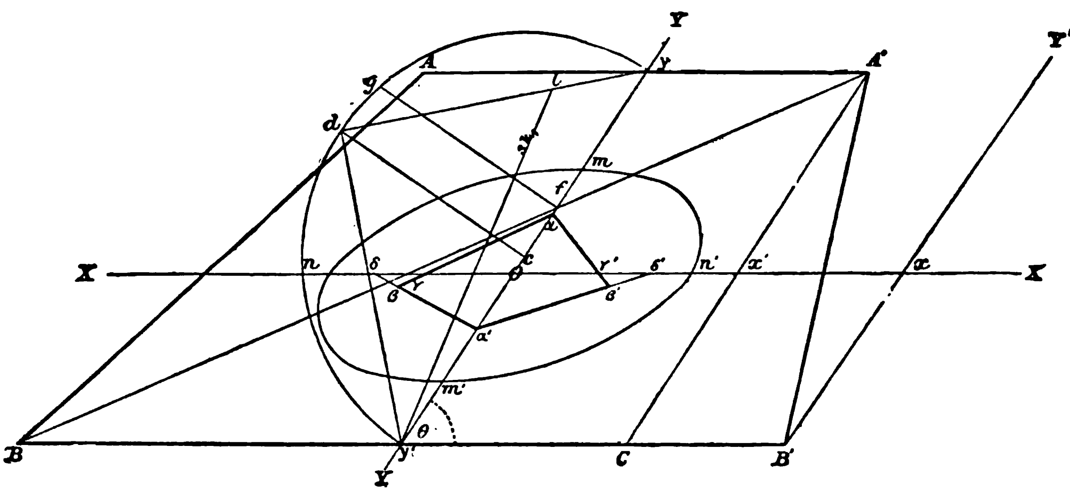
The kern can therefore be drawn at once without the aid of the central ellipse. From the kern any number of points on the latter can be obtained. Thus $O m^2 = O \alpha \cdot O y$, and $O n^2 = O \beta \cdot O x$. The diagonals A′ A, B′ B are the directions of conjugate axes of the ellipse, and if A′ A cuts the side $\alpha \beta$ of the kern in ϵ, then $O t^2 = O \epsilon \cdot O A'$. Eight points on the ellipse or two pairs of conjugate diameters are thus obtained. The curve could be drawn from either pair of diameters, or other points could be obtained by means of its geometrical properties.

Rectangle.—For the rectangle $\sin \phi = 1$ and k_1, k_2, determined in precisely the same way, are the semi-major and minor axes of the central ellipse. The kern is in this case a rhombus whose diagonals are respectively one-third of the sides of the rectangle parallel to them.

Square.—For the square $\sin \phi = 1$ and $a = b$. The central ellipse becomes a circle whose radius is $a \sqrt{\frac{1}{12}}$ where a is the side of the square. The kern is a square whose diagonals are parallel to and one-third of the sides of the square.

Trapezium.—In Fig. 110—X X and Y Y drawn through O, the centre of gravity of the figure, the former parallel to, the latter bisecting its parallel sides A A′, B B′, are the directions of two conjugate diameters of the central ellipse.

FIG. 110.



The moment of inertia about X X (if the lever arms are taken parallel to Y Y) is (see Table, p. 119.)

$$\frac{h^3}{36} \cdot \frac{b^2 + 4 b b_1 + b_1^2}{b + b_1} \cdot \sin \theta.$$

The area is—

$$\frac{h}{2} (b + b_1) \sin \theta,$$

where

$$y y' = h.$$

Hence, if k_1 is the radius of gyration about X X—

$$k_1^2 = \frac{h^2}{18} \cdot \frac{b^2 + 4 b b_1 + b_1^2}{(b+b_1)^2} = h^2 \left[\frac{1}{18} + \frac{b b_1}{9 (b+b_1)^2} \right].$$

To obtain k_1 by construction, since

$$(3\,k_1)^2 = \tfrac{1}{2}\,h^2 + \frac{b\,b_1}{(b+b_1)^2}\,h^2,$$

describe a semicircle on $y\,y'$ cutting a perpendicular to $y\,y$ drawn from e its middle point in d, and a second perpendicular drawn from f, the point where the diagonal A′ B cuts $y\,y$, in g. Then—

$$\overline{y\,d}^2 = \tfrac{1}{2}\,h^2 \text{ and } \overline{f\,g}^2 = y\,f \,.\, y'\,f = \frac{b\,b_1}{(b+b_1)^2}\,.\,h^2.$$

$$y\,f = \frac{b_1}{b+b_1}\,.\,h \text{ and } y'\,f = \frac{b}{b+b_1}\,.\,h.$$

Make $d\,l$ equal to $f\,g$. Then—

$$y'\,l = \sqrt{\tfrac{1}{2}\,h^2 + \frac{b\,b_1}{(b+b_1)^2}\,.\,h^2} = 3\,k_1.$$

Set off O m, O m' from O along Y Y equal to k_1, then $m\,m'$ is the diameter of the ellipse conjugate to the direction of X X.

The moment of inertia of the trapezium about Y Y is

$$\frac{h}{48}(b^3 + b^2\,b_1 + b\,b_1^2 + b_1^3)\sin\theta.$$

Hence
$$k_2^2 = \tfrac{1}{24}\,(b^2 + b_1^2),$$

which can easily be obtained by construction.

The kern can be determined by means of the central ellipse by drawing in the latter the polars of points symmetrical to A, A′, B, B′, with respect to O: it can also be obtained directly from the radii of gyration about X X and Y Y, as follows. Suppose the axis to coincide with B B′, then a the stress centre lies on Y Y the direction conjugate to B B′ in the central ellipse, and

$$O\,a = \frac{k_1^2}{O\,y'}. \quad \text{Similarly } O\,a' = \frac{k_1^2}{O\,y}.$$

When the axis turns about B′ into a position B′ Y′ parallel to Y Y², the centre of moments is on X X at a point γ such that

$$O\,\gamma = \frac{k_2^2}{O\,x}.$$

Suppose that the axis comes into the position A′ C parallel to Y Y cutting X X in x'. Then if

$$O\,\delta = \frac{k_2^2}{O\,x'},$$

δ is a point on the produced *locus* of the stress centre when the axis turns about A′ The corresponding points $\gamma'\,\delta'$ are obviously symmetrical to γ and δ with respect to O. Join $a\,\gamma$, $a\,\gamma'$ and produce them to cut $a'\,\delta$, $a'\,\delta'$ in β and β': Then $a\,\beta\,a'\,\beta'$ is the kern of the trapezium. Points on the central ellipse can now be obtained from the kern.

Triangle.—In Fig. 113. A D bisecting B C and X X through O (the centre of gravity of the triangle) parallel to B C are conjugate axes of the central ellipse. The moment of inertia about X X (the lever arms being taken parallel to Y Y) is $\frac{b\,h^3}{36}\sin\phi$, where $h = AD$; $b = BC$. Hence

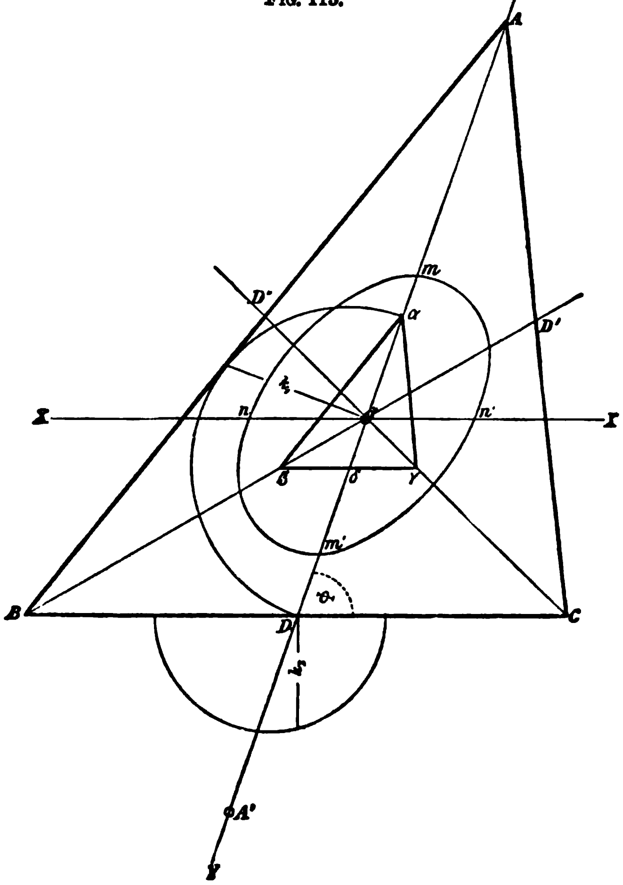
$$k_1^2 = \frac{\frac{b\,h^3 \sin\phi}{36}}{\frac{b\,h \sin\phi}{2}} = \frac{1}{18}h$$

$$k_1 = \sqrt{\frac{h}{6}\cdot\frac{h}{3}}$$

make $O\,m = O\,m' = k_1$.

The moment of inertia of the triangle A B D about Y Y is $\frac{h}{12}\left(\frac{b}{2}\right)$, $\sin\phi$, where

FIG. 113.



$AD = h$ and $BC = b$. The moment of inertia of the whole triangle about Y Y is therefore $\frac{h}{2}\left(\frac{b}{6}\right)^3 \sin\phi$ and—

$$k_2^2 = \frac{\frac{h}{6}\left(\frac{b}{2}\right)^2 \sin\phi}{\frac{h\,b}{2}\sin\phi} = \frac{b^2}{24}.$$

Thus $$k_2 = \sqrt{\tfrac{1}{3} \cdot \frac{b}{2} \cdot \tfrac{1}{3} \cdot \frac{b}{2}} = \sqrt{\frac{\mathrm{BD}}{2} \cdot \frac{\mathrm{BD}}{3}}$$

make $\mathrm{O}n = \mathrm{O}n' = k_2$. A pair of conjugate diameters $m\,m'$, $n\,n'$ are thus obtained. Drawing axes parallel to A C, A B, and proceeding as before, two new pairs of conjugate diameters can be determined, thus giving twelve points on the ellipse. Or, the latter can be drawn at once by means of the conjugate axes $m\,m'$, $n\,n'$.

For the kern, suppose the axis to coincide with B C; then the stress centre a lies on Y Y and is in the central ellipse the pole of a line symmetrical to B C with respect to O. Hence—

$$\mathrm{O}\,a = \frac{k_1^2}{\mathrm{O\,D}} = \frac{\frac{h^2}{18}}{\frac{h}{3}} = \tfrac{1}{6}\,h.$$

Similarly $$\mathrm{O}\,\beta = \tfrac{1}{6}\,\mathrm{B\,D}'; \quad \mathrm{O}\,\gamma = \tfrac{1}{6} \cdot \mathrm{C\,D}''.$$

Join $a\,\beta$, $\beta\,\gamma$, $\gamma\,a$; then $a\,\beta\,\gamma$ is evidently a similar, similarly situated triangle to A B C, whose sides are respectively one-fourth of the corresponding sides of A B C.

Again, suppose the axis to turn about A then $\beta\,\gamma$ is the polar of a point A′ on Y Y symmetrical to A with respect to O, and

$$\mathrm{O}\,\delta = \frac{k_1^2}{\mathrm{O\,A}'}.$$

But $$\mathrm{O}\,\delta = \frac{h}{12} \text{ and } \mathrm{OA}' = \tfrac{2}{3}\,h.$$

Hence as before $k_1 = \sqrt{\frac{h}{6} \cdot \frac{h}{3}}$. Thus, if the kern is first drawn, the ellipse can be obtained.

Isosceles Triangle.—If A B = A C the diameters $m\,m'$, $n\,n'$ become the principal axes of the central ellipse.

Equilateral Triangle.—If A B = B C = A C, then $h = \sqrt{\frac{3}{2}}\,b$. Thus

$$k_1^2 = \frac{b^2}{24} = k_2^2.$$

The central ellipse therefore becomes a circle whose radius is $b\sqrt{\frac{1}{24}}$ where b is the side of the triangle. The kern is a similarly situated equilateral triangle whose sides are equal to $\frac{b}{4}$.

Circle.—The moment of inertia about an axis through the centre is $\frac{\pi}{64}\,d^4$, where d is the diameter. The radius of gyration about any axis is therefore—

$$\sqrt{\frac{\frac{\pi}{64}\,d^4}{\pi\left(\frac{d}{2}\right)^2}} = \frac{d}{4}.$$

The central ellipse is therefore a concentric circle whose diameter is $\frac{d}{2}$.

The kern is evidently another concentric circle, and its radius

$$r = \frac{\left(\frac{d}{4}\right)^2}{\frac{d}{2}} = \frac{d}{8}.$$

Ellipse.—Draw two conjugate diameters A A′, B B′. Let A A′ = $2a$; B B′ = $2b$ and θ = the included angle. The moment of inertia about A A′ is

$$\frac{\pi}{4} a b^3 . \sin\theta,$$

the area is $\pi a b \sin\theta$. Hence if k_1 is the radius of gyration about A A′

$$k_1^2 = \frac{b^2}{4} \text{ and } k_1 = \frac{b}{2},$$

similarly for the other axis B B′ $\quad k_2 = \frac{a}{2}.$

The central ellipse is therefore a similar concentric ellipse whose axes are half the length of those of the ellipse itself.

When the axis touches the ellipse at A, the position of a the stress centre is on A A′, and if O is the centre of the ellipse

$$\text{O}a = \frac{k_2^2}{a} = \frac{a}{4}.$$

The kern is therefore another similar concentric ellipse whose axes are one-fourth of those of the original figure.

83. **Central Ellipse and Kern of an I Section.**—The section (Fig. 111, Pl. X.) is symmetrical about X X and Y Y. These lines of symmetry are therefore the directions of the principal axes of the central ellipse, and their intersection G is its centre.

To determine the moment of inertia about X X, the section is cut up into 10 laminas by lines parallel to XX. Since the section is symmetrical about X X, it will only be necessary to deal with the 5 laminas $A_1 \ldots A_5$ forming its upper half. Of these A_1 and A_5 are rectangles; A_2, A_3, A_4 approximately trapeziums. Set off the areas $A_1 \ldots A_5$ (reduced to any common base) along a load line $\overline{0\,5}$. In the present instance the areas were found by scaling off the dimensions and multiplying out. Through $g_1, g_2 \ldots$, the centres of gravity of the laminas, draw lines parallel to X X. Consider the areas A_1, $A_2 \ldots$ to act as parallel forces along these lines, and draw the funicular polygon I... V of these forces with respect to a pole O at a distance H from the load line $\overline{0\,5}$. Produce the sides of this polygon to cut X X in 0′ 1′.. 5′ respectively. Then (§ 68) $\overline{0'1'}$, $\overline{1'2'}$... are the reduced moments about X X of A_1, $A_2 \ldots$ These reduced moments must now be supposed to act as parallel forces at the stress centres with respect to X X of each lamina; e. g. $\overline{4'5'}$ must be taken to act at the stress centre with respect to X X of all the elementary areas which go to make up A_5. Now g_5 is the centre of the central ellipse of the area A_5, and if m_5 (obtained as in § 82) is its semi-axis on Y Y, then the distance of the required stress centre (a_5) from g_5 is equal to $\frac{(m_5)^2}{g_5 G}$ (§ 72–4). The position of a_5 can, therefore, be obtained by construction. Draw a line through a_5 parallel to X X. In the case of the remaining areas, $A_1 \ldots A_4$, the corresponding semi-axes $m_1 \ldots m_4$ are very small relatively to the distances $g_1 G$, $g_2 G$, ... : the distances $a_1 g_1$, $a_2 g_2$ . . are very nearly *nil*, and the reduced moments $0'1' \ldots 3'4'$ may be taken to act at $g_1 \ldots g_4$.

Using $0'5'$ as a load line, draw the funicular polygon I′. . . V′ with a polar distance H′. The first and last sides of this polygon cut X X in 0″, 5″. Then (§ 68) the moment of inertia of the upper half of the section about X X is—

$$H \,.\, H' \,.\, \overline{0''5''},$$

and the moment of inertia of the whole section is—

$$2H \,.\, H' \,.\, \overline{0''5''}.$$

The radius of gyration (k_1) about X X is—

$$\sqrt{\frac{2H \,.\, H' \,.\, 0''5''}{2(A_1 + A_2 \ldots + A_5)}} = \sqrt{\frac{H \,.\, H' \,.\, 0''5''}{\overline{0\,5}}}$$

But H′ was taken equal to $\overline{0\,5}$.

Hence— $$k_1 = \sqrt{H \,.\, \overline{0''5''}}.$$

Make G m and G m' equal to k_1. Then $m\,m'$ is the principal axis of the central ellipse lying on Y Y.

For the moment of inertia about Y Y, the same process might be followed, the section being cut up into laminas by lines parallel to Y Y. It will be simpler however, to make use of the original laminas, and to adopt the construction indicated in § 73. Obtain b_1, b_2 . . . , the extremities of the axes parallel to X X of the central ellipses of the several laminas, and suppose the areas A_1, A_2 . . . to act as forces parallel to Y Y at b_1, b_2 . . . respectively. Between parallels to Y Y from b_1, b_2, . . . draw the funicular polygon $I_1 \ldots V_1$ of the forces A_1, A_2 . . .* It is not necessary to draw a new load line parallel to Y Y, the sides of the funicular polygon are merely drawn respectively at right angles to the vectors 0 O″, 1 O″. . . . . A shorter polar distance H″ has been taken, in order to spread out the funicular polygon somewhat. The intercepts $\overline{0_1 1_1}$, $\overline{1_1 2_1}$ . . . on Y Y, given by the produced sides of the polygons, are now treated as parallel forces acting also at b_1, b_2. . . . A second funicular polygon $I_2 \ldots V_2$

* In the figure the parallel to Y Y from b_4 coincides with the outline of the section.

with a pole O‴ at a distance H‴ from the load line $\overline{0_1 5_1}$ is now drawn. Its first and last sides cut Y Y in 0_2, 5_2. Then k_2 the radius of gyration of the whole section about Y Y is equal to—

$$\sqrt{\frac{H''.\,H'''.\,\overline{0_2 5_2}}{\overline{0\,5}}}.$$

In the figure H″ = H‴ = ½ . $\overline{0\,5}$.

Hence—

$$k_2 = \sqrt{\frac{H''}{2}\,.\,\overline{0_2 5_2}}.$$

Set off G *n*, G *n′* each equal to k_2 from G along X X; then *n n′* is the other axis of the central ellipse.

The kern is evidently a rhombus whose angles lie on X X and Y Y. When the axis coincides with the lowest layer of fibres $D_1 D_2$, the centre of moments *a* lies on Y Y at a distance from G equal to

$$\frac{\overline{G\,m}^2}{G\,D} = \frac{k_1^2}{G\,D}.$$

This determines the position of *a*.

When the axis is in the position $E_1 E_1$, the stress centre β lies on X X at a distance from G equal to $\frac{k_2^2}{G\,E}$; *a′* and β' are respectively symmetrical to *a* and β. The line $a\,\beta$ is the *locus* of the stress centre when the axis turns about the point D_2.

The central ellipse of such a section having been drawn once for all, the moment of resistance (M) of the section about any axis whatever can be immediately deduced. For (§ 75) $M = \frac{f_p}{d_p}\,.\,I$; or $\frac{f_t}{d_t}\,.\,I$, and $I = A\,K^2$, where A is the area of the section and K the radius of gyration. From the central ellipse the value of K for *any axis passing through* G (Fig. 111, Pl. X.) can be obtained (§ 71). The radius of gyration (*k*) about any parallel axis in the same plane distant D from the first can thence be deduced, for $k^2 = K^2 + D^2$ (§ 70).

84. **Central Ellipse and Kern of an Angle Iron.**—Fig. 112, Pl. XI. The angle iron is dealt with merely as two rectangles, the rounding off of the angles being neglected. This rounding off can be approximately taken into account by considering the ends of the section as trapeziums, and the filling up of the angle as a triangle; very little difference is, however, occasioned by neglecting it altogether, and the construction is simplified.

The centre of gravity G of the whole section divides $g_1\,g_2$, the line joining the centres of gravity of its two component rectangles, inversely as the areas of the latter. Through G draw an axis X X parallel to one of the arms of the section. Set off 0 1, 1 2, representing the areas of the rectangles, along a load line parallel to X X. Take a pole O at a distance H from 0 2. Draw lines through g_1, g_2 parallel to X X, and place the funicular polygon 0′ I II between them. The first and last sides of this polygon intersect on X X, since X X passes through the centre of gravity: the reduced moments of the two rectangles about X X are equal, and equal to the intercept 0′ 1′. It is necessary now to find the stress centres of the rectangles with respect to X X, and to suppose the reduced moments to act as parallel forces at these stress centres. The principal axes of the central ellipses of the rectangles are obtained (§ 82); then, if a is the semi-major axis of the central ellipse of the larger rectangle and d the distance of g_1 from X X—

$$\overline{g_1 g^1} = \frac{a_2}{d}$$

which determines the position of g', the stress centre of the first rectangle. The required stress centre of the other rectangle is similarly obtained; it lies at a very short distance from g_2. From these stress centres draw lines parallel to X X, and taking a pole O′ at a distance H′ from the new load line 1′ 0′, draw the funicular polygon 0″ I′ II′ 2″. Then (§ 70) k_1 the radius of gyration of the whole figure about X X is equal to $\sqrt{\dfrac{H \cdot H' \cdot \overline{0'' 2''}}{\overline{0\,2}}}$; and since H′ has been taken equal to half $\overline{0\,2}$—

$$k_1 = \sqrt{\frac{H}{2} \cdot \overline{0'' 2''}}.$$

We must now either draw any two axes through G, and obtain the radii of gyration about each separately, or determine the direction of Y Y, the axis conjugate to X X, and obtain the radius of gyration about Y Y: the latter course is here followed. The line joining the two stress centres with respect to X X is (§ 72–3) the direction conjugate to X X. Draw Y Y through G parallel to this line. To find the radius of gyration about Y Y, the method of § 73 may be employed. Thus, draw tangents to the central ellipses of the rectangles parallel to Y Y. Then if r_1 and r_2 are the distances of g_1, g_2 from the respective tangents and d_1, d_2, the distances of g_1, g_2 from YY, draw lines parallel to Y Y at distances from it equal to $\sqrt{d_1^2 + r_1^2}$ and $\sqrt{d_2^2 + r_2^2}$, and on the same side of Y Y as g_1 and g_2 respectively: the construction is shown. Between these lines draw the funicular polygon $0_1'$ I_1 II_1 $2_1'$, its sides being respectively parallel to the vectors 0_1 O_1, 1_1 O_1, 2_1 O_1, of a new load line 0_1 2_1, parallel to Y Y. The intercepts $0_1'$ $1_1'$, $1_1'$ $2_1'$, of the sides of this funicular polygon on Y Y are the reduced moments about Y Y of the two rectangles. Suppose these reduced moments to act as parallel forces along the same lines, and draw the funicular polygon 0_2 I_2 II_2 2_2 with respect to the pole O_2. Then k_2, the radius of gyration of the whole section about Y Y, is equal to $\sqrt{\dfrac{H_1 \cdot H_2\,\overline{0_2\,2_2}}{0_1\,2_1}}$; or, since H_2 has been taken equal to half $\overline{0_1\,2_1}$—

$$k_2 = \sqrt{\frac{H_1}{2} \cdot \overline{0_2 2_2}},$$

whence k_2 is obtained. Lines parallel to and distant k_2 from Y Y cut X X in the extremities of one axis of the required central ellipse of the section. Lines parallel to and distant k_1 from X X cut Y Y in the extremities of the conjugate axis. From these two conjugate axes the central ellipse is drawn.

The kern is a five-sided figure, the construction of which presents no difficulty. When the axis coincides with A B, the stress centre lines on Y Y, at a distance G α from G equal to $\frac{\overline{G e}^2}{G f}$. When the axis coincides with C D, the stress centre is at a point β on Y Y at a distance from G equal to $\frac{\overline{G e}^2}{G e'}$. When the axis takes the position R S, the stress centre is at γ, a point on the axis of the ellipse conjugate to the direction of R S, such that $G \gamma = \frac{\overline{G m}^2}{G m'}$. The point δ on the same axis of the ellipse, obtained in the same way, is the position of the stress centre for an axis parallel to R S, and passing through A. The points ϵ and θ, found in the same way, are the positions of the stress centre for vertical axes through B and A respectively.

85. **Resistance to Shearing.**—In order that a section may be able to resist the action of shearing force, it is necessary that the maximum intensity of shearing stress on any layer of fibres of the section should not exceed the safe resistance to shearing of the material. Shearing stress is not uniformly distributed over a section, but may be taken as uniform along lines parallel to the neutral axis of the section.

If V is the resultant shearing stress which a section of area A can safely resist; then—

$$V = \tfrac{4}{5} A \, . \, p; \text{ or, } V = A \, . \, s,$$

where p is the *less* of the safe stresses (per unit of area) of the material in tension and pressure, and s is the safe load in shear.*

86. **Intensity of Stress at the Neutral Axis.**—If $\alpha\,\alpha$, $\beta\,\beta$ (Fig. 114) are two cross sections of a beam distant z apart, and P_1, P_2 are the resultant stresses in all the fibres above the neutral surface of the beam at $\alpha\,\alpha$ and $\beta\,\beta$ respectively, then the mean shearing stress over the layer of fibres $\alpha\,\alpha'$ at the neutral surface is equal to $P_1 - P_2$, and the intensity of stress is $\frac{P_1 - P_2}{\text{area of layer}}$. This shearing stress approximates more nearly to the tangential, or vertical shearing stress at the neutral axis of the cross section at $\alpha\,\alpha$, as the distance z diminishes.

For beams of simple section, the magnitude of this shearing stress may be ascertained from a consideration of the resistance areas at two adjacent cross sections. Thus, suppose the beam (Fig. 114) to have a uniform rectangular section—height h,

* See Appendix.

breadth b. Let R be the reaction at B, l the distance from B to the cross section $a\,a$; M_1 and M_2 the bending moments at $a\,a$, $\beta\,\beta$ respectively, and A_1 and A_2 the areas of the corresponding resistance areas. Then (§ 76)—

$$M_1 = \tfrac{2}{3} h \,.\, A_1 \text{ and } M_2 = \tfrac{2}{3} h \,.\, A_2,$$

but—

$$M_1 = R \,.\, l \text{ and } M_2 = R\,(l - z).$$

Hence—

$$A_1 = \tfrac{3}{2}\frac{R \,.\, l}{h} \text{ and } A_2 = \tfrac{3}{2}\frac{R \,.\, l}{h} - \tfrac{3}{2}\frac{R \,.\, z}{h}.$$

Now—

$$P_1 - P_2 = A_1 - A_2 = \tfrac{3}{2}\frac{R \,.\, z}{h} = V$$

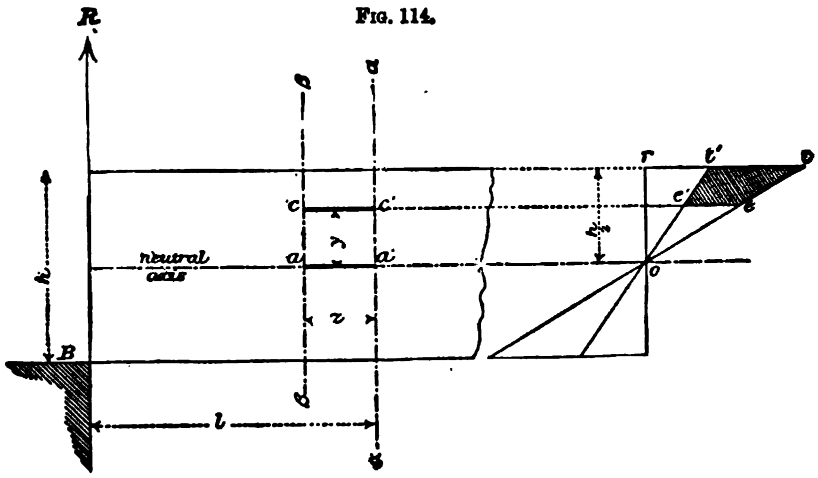
the total shearing stress on $a\,a'$. The *intensity* of shearing stress (v) on $a\,a'$ is equal to $\frac{V}{b \,.\, z}$.

Thus—

$$v = \tfrac{3}{2}\frac{R}{b\,h}.$$

The total shearing stress on the section $a\,a$ is equal to R, and the *mean intensity* of

FIG. 114.



shearing stress is $\frac{R}{b\,h}$. Hence the ratio of the intensity of shearing stress at the neutral axis of the section $a\,a$ to the mean intensity of shearing stress over the whole section is $\frac{3}{2}:1$. For a rectangular section the intensity of shearing stress is greatest at the neutral axis.

87. **Intensity of Stress at any part of a Section.**—The shearing stress on any layer of fibre $c\,c'$ (Fig. 114) parallel to the neutral surface of the beam is equal to the difference between the resultant pressures at $a\,a$ and $\beta\,\beta$ of all the fibres above $c\,c'$

Let Y_1, Y_2 be the vertical ordinates under $\alpha\,\alpha$, $\beta\,\beta$ of the curve of bending moments; then, if H is the polar distance—

$$M_1 = Y_1 \,.\, H \ (\S\, 44).$$

Hence—

$$A_1 = \frac{3}{2\,h} \,.\, Y_1 \,.\, H,$$

and similarly—

$$A_2 = \frac{3}{2\,h} Y_2 \,.\, H.$$

Thus, if $r\,t$ is made equal to $\frac{4\,A_1}{h}$ and $r\,t'$ equal to $\frac{4\,A_2}{h}$, the triangles $r\,t\,o$, $r\,t'\,o$ are the resistance areas corresponding to M_1 and M_2 for the sections $\alpha\,\alpha$, $\beta\,\beta$. Produce $c\,c'$ to cut $t'\,o$, $t\,o$ in e', e: then the cross-lined figure $t\,t'\,e'\,e$ represents the total shearing stress on $c\,c'$.

From the above considerations a general expression for the intensity of shearing stress on any layer of fibres parallel to the neutral axis in a section of any form can be directly deduced. Let f_1 be the intensity of direct stress (tension, or pressure) on a horizontal layer of fibres of length x distant y from the neutral axis of any section. Let V be the shearing force, M_1 the bending moment at this section, and l the distance of the section from the support of the beam. Let f_2 and M_2 be the corresponding stress and bending moment for a second section distant z from the first: then—

$$M_1 = V \,.\, l$$

$$M_2 = V\,(l - z).$$

Now (§ 75)

$$f_1 = \frac{M_1 \,.\, y}{I} \text{ and } f_2 = \frac{M_2 \,.\, y}{I},$$

where I is the moment of inertia of the whole section about the neutral axis. Since the two sections are supposed very near, the value of I is the same for both. The resultant of all the direct stresses in the fibres above the layer x in the first section is

$$\Sigma_y^{h'} f_1 \,.\, x\,y = \frac{M_1}{I} \Sigma_y^{h'} x\,y$$

where h' is the distance of the uppermost layer of fibres from the neutral axis. The corresponding resultant for the second section is similarly

$$\frac{M_2}{I} \Sigma_y^{h'} x\,y.$$

Hence the total shearing stress for the layer x of the first section is, if z is very small,

$$\frac{1}{I} \Sigma_y^{h'} x\,y\,(M_1 - M_2).$$

But

$$M_1 - M_2 = V\,z.$$

Hence the required shearing stress is equal to

$$\frac{Vz}{I}\Sigma_y^{h'} x y,$$

and its intensity (v) is

$$\frac{V \, . \, z}{I \, . \, z \, . \, x}\Sigma_y^{h'} x y = \frac{V}{I \, . \, x}\Sigma_y^{h'} x y;$$

or, expressed in the symbols of the Integral Calculus,

$$v = \frac{V}{I x}\int_y^{h'} x y \, d y \quad . \; . \; . \; . \; . \; . \; . \quad (a)$$

The intensity of shearing stress is a maximum for a layer of fibres at a distance y from the neutral axis such that

$$\frac{1}{x}\int_y^{h'} x y \, d y$$

is a maximum.

The following* are the intensities of shearing stress for a few simple sections calculated from equation (a).

Rectangle.—Height h, breadth b. (No. I., p. 118.)

$$v = \frac{3}{2}\frac{V}{b\,h}\left[1 - \left(\frac{y}{h'}\right)^2\right]; \; h' = \frac{h}{2}.$$

$$v \text{ max.} = \frac{3}{2}\frac{V}{b\,h} \text{ for } y = 0.$$

Circle.—Radius r. (No. XIV., p. 119.)

$$v = \frac{4}{3\,\pi\,r^2}\left(1 - \frac{y^2}{r^2}\right); \; h' = r.$$

$$v \text{ max.} = \frac{4}{3\,\pi\,r^2} \text{ for } y = 0.$$

Square on edge.—Side b. (No. III., p. 118.)

$$v = \frac{V}{b^2}\left[1 + \frac{y}{h'} - 2\left(\frac{y}{h'}\right)^2\right]; \; h' = \frac{b}{\sqrt{2}}.$$

$$v \text{ max.} = \frac{9\,V}{8\,b^2} \text{ for } y = \frac{h'}{4}.$$

I *Section.* (No. VI., p. 118.)

$$v \text{ max.} = \frac{3}{2} \, . \, \frac{V}{b_1} \, . \, \frac{b\,h^2 - (b - b_1)\,h_1^2}{b\,h^3 - (b - b_1)\,h_1^3} \text{ for } y = o.$$

88. **Curve of Shearing Stress.** — A curve representing the distribution of shearing stress over any section can be drawn by the following method. Taking first the rectangular section (Fig. 115), construct the equivalent area $r\,t\,o$ in the usual way

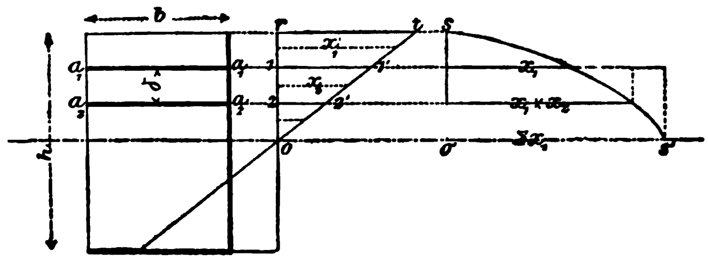
* 'Taschenbuch der Hütte.'

(§ 76). Draw the equidistant horizontal ordinates $1\,1'$, $2\,2'$, ... cutting the area up into strips of equal depth δ. Let $y_1, y_2 \ldots$ be the distances of $1\,1'$, $2\,2' \ldots$ from the neutral axis: then

$$\frac{\overline{11'}}{y} = \frac{b}{\frac{h}{2}} \text{ and } \Sigma(\overline{11'}) = \frac{2}{h}\Sigma(b\,y).$$

Now the shearing stress for the layer of fibres $a_1 a_1$ is proportional to $\Sigma_{y_1}^{\frac{h}{2}} b\,y$, and the area $r\,t\,1'\,1$ above $1\,1'$ is equal to $\frac{2}{h}\Sigma_{y_1}^{\frac{h}{2}} b\,y$; similarly the area $r\,t\,2'\,2$ represents the shearing stress at the layer $a_2 a_2$, and so on. Hence it is only necessary to construct a

Fig. 115.



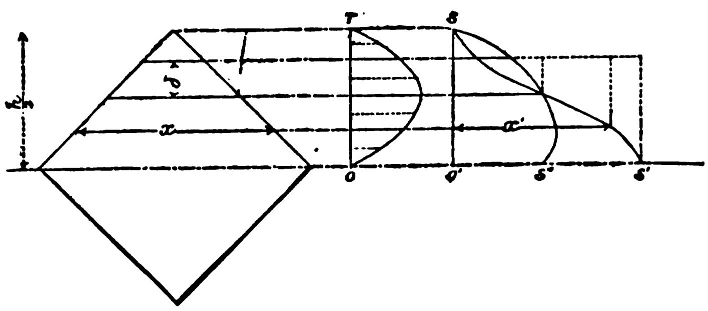
curve whose ordinates are respectively proportional to the *areas* of the portions of the equivalent areas above them. Let x_1, $x_2 \ldots$ be the mean breadths of the trapeziums $r\,1'$, $1\,2' \ldots$. From any vertical axis $s\,o'$ set off x_1 on $1\,1'$ produced, $x_1 + x_2$ on $2\,2'$ produced, and so on; finally, set off $\Sigma(x_1)$ on the neutral axis. The curve $s\,s'$ thus obtained gives the distribution of shearing stress on the section. Thus the ordinate of this curve multiplied by $\frac{V}{I} \cdot \delta \cdot \frac{h}{2}$ gives the magnitude of the shearing stress on any horizontal layer of fibres, and multiplied by $\frac{V}{I} \cdot \frac{\delta}{b} \cdot \frac{h}{2}$ gives the *intensity* of shearing stress on the same layer. Moreover $I = \frac{2}{3} h \cdot \overline{o's'} \cdot \delta \cdot \frac{h}{2}$. Hence the intensity on any layer is given by the corresponding ordinate of $s\,s'$ multiplied by $\frac{3\,V}{b \cdot h \cdot o'\,s'}$. For a section symmetrical about the neutral axis, the curve of shearing stress distribution is, of course, symmetrical also. If the ordinate $o'\,s'$ $(= \Sigma\,x_1)$ is inconveniently large, the distances x_1, $x_1 + x_2$, &c., should be plotted to a scale of $\frac{1}{2}$ or $\frac{1}{4}$. It follows from the above that the total shearing stress is always a maximum at the neutral axis, although the maximum *intensity* may occur at quite another position.

The curve $s\,s'$ (Fig. 116), obtained from the parabolic equivalent area in precisely the same way, gives the distribution of shearing stress on the square section shown: the intensity of shearing stress (v) on any layer of fibres x of this section is equal to

$\frac{x'}{x}\cdot\frac{V\cdot}{I}\cdot\frac{h}{2}\cdot\delta$, where x' is the ordinate of the curve $s\,s'$ corresponding to x. In this case $I = \frac{h}{2}\cdot o'\,s'\cdot\delta\cdot\frac{h}{2}$. Hence $v = \frac{x'}{x}\cdot\frac{2V}{h\,.\,o's'}$.

A curve can easily be drawn representing the intensity of stress at all parts of the

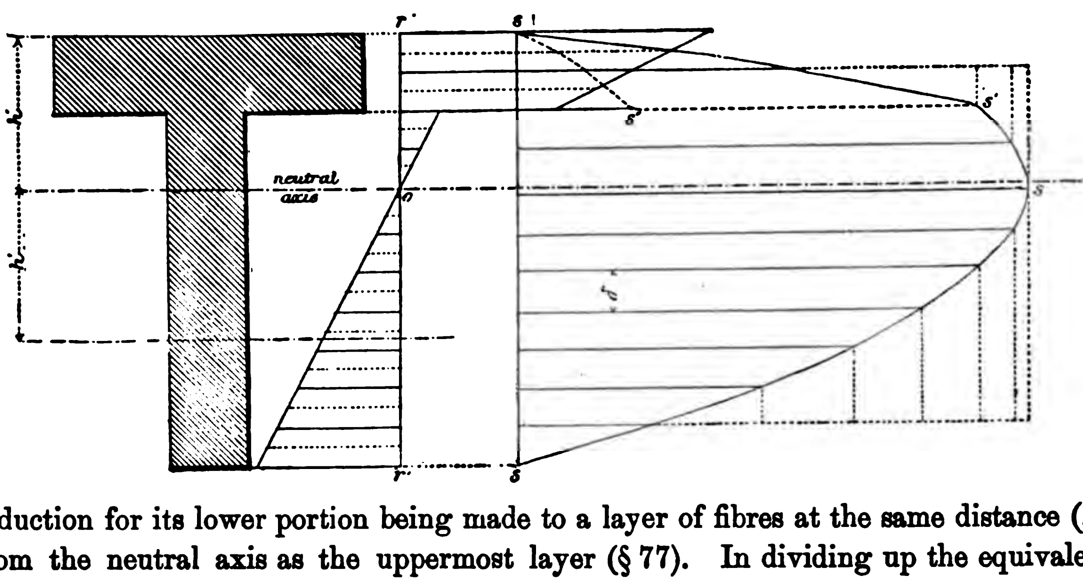
FIG. 116.



section. If x'' is the ordinate of the new curve corresponding to x', then $x'' = \frac{x'}{x}$. Thus by constructing $\frac{x''}{C} = \frac{x'}{x}$, where C is a constant, any number of points on the intensity curve $s\,s''$ are obtained. The constant C should be chosen so as to bring the ordinates of $s\,s''$ within convenient limits. In drawing the curve $s\,s''$, the constant C was taken equal to $\frac{h}{2}$, and the two curves have a common ordinate at a distance $\frac{h}{4}$ from the neutral axis.

For the T section (Fig. 117) the equivalent area is drawn in the usual way, the

FIG. 117.



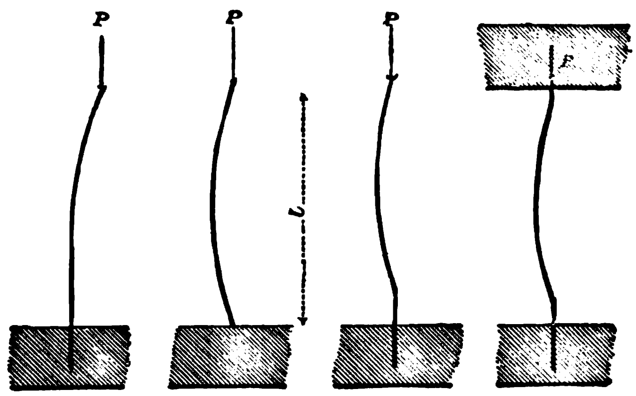
reduction for its lower portion being made to a layer of fibres at the same distance (h') from the neutral axis as the uppermost layer (§ 77). In dividing up the equivalent area into strips, it will be most convenient to make δ the height of a strip a simple fraction—in the figure $\frac{1}{2}$—of the thickness of the flange of the section, setting this distance off on $r r'$, from r down and from r' up, as far as the neutral axis. The distribution curve $s s' s s$ is drawn as before. In deducing the intensity curve, the constant C has been taken equal to the breadth of the web of the T section: the portion $s' s s$ of this curve, therefore, coincides with the distribution curve, while the upper portion takes the form $s s'' s'$ as shown by the dotted line. In a section of this form the maximum intensity of shearing stress obviously occurs at the neutral axis.

89. **Resistance of Long Struts.**—A pressure bar or strut, if the ratio of its length to its least cross dimensions exceeds a fixed value depending both on the material and on the form of the section, tends to fail by cross-breaking. Cross-breaking involves a combination of bending and direct stress in the bar, which may cause the latter to fail before the thrust it has to sustain reaches the safe load in simple pressure. Various empirical formulæ have been proposed for the resistance of long struts, but the experiments on which they are based are few in number, and the formulæ themselves are not free from objection. The rational formula alone is therefore given, which, moreover, has the advantage of being easy to apply.

The following four cases arise (Fig. 118):—

I. Strut fixed at one end and free at the other.
II. Both ends free, but guided in the direction of the thrust.
III. One end fixed, the other free, but guided in the direction of the thrust.
IV. Both ends fixed.

FIG. 118.



The corresponding formulæ are—

$$\text{I. } P = n \cdot \frac{\pi^2}{4} \cdot \frac{E \cdot I}{l^2} \quad \ldots\ldots \quad (\alpha)$$

$$\text{II. } P = n \cdot \pi^2 \frac{E I}{l^2} \quad \ldots\ldots \quad (\beta)$$

$$\text{III. } P = 2n\pi^2\frac{EI}{l^2} \quad . \; . \; . \; . \; . \; . \quad (\gamma)$$

$$\text{IV. } P = 4n\pi^2\frac{EI}{l^2} \quad . \; . \; . \; . \; . \; . \quad (\delta)$$

Where—

P is the safe load.
n the coefficient of safety.
l the length of the strut.
E the modulus of elasticity.*
I the least moment of inertia of the section about an axis through the centre of gravity. (If A is the area of the section, then $I = A \, . \, k^2$, where k is the semi-minor axis of the central ellipse of the section.)

The values assigned to n may be—

For wrought iron	$\frac{1}{5}$
" cast "	$\frac{1}{8}$
" wood	$\frac{1}{10}$ to $\frac{1}{12}$

The formulæ (α), (β), (γ), (δ) are to be applied only when $\frac{l}{h}$, or $\frac{l}{d}$ exceed the following values, h being the shortest side of a rectangular section, d the diameter of a circular section.

Material.	$\frac{l}{h}$	$\frac{l}{d}$	Formula.
Wrought iron	14	12	(α)
Cast "	$5\frac{3}{4}$	5	
Wood	8	6	
Wrought iron	28	24	(β)
Cast "	$11\frac{1}{2}$	10	
Wood	$13\frac{1}{2}$	$11\frac{1}{2}$	
Wrought iron	38	33	(γ)
Cast "	16	14	
Wood	19	16	
Wrought iron	56	48	(δ)
Cast "	23	20	
Wood	27	23	

* See Appendix.

APPENDIX.

WEIGHT OF A CUBIC FOOT OF VARIOUS SUBSTANCES.

Substance.	Lb.	Tons.
Wrought Iron	480	·2143
Cast Iron	450	·2009
Steel	487	·2174
Oak	52	·0232
Elm	35	·0156
Red Pine	37·5	·0167
Masonry	130	·0580
Brickwork	112	·0500
Concrete	120	·0536
Earth	100	·0446
Water	62·5	·0279

ROOFS.

WEIGHT OF VARIOUS ROOF COVERINGS IN LB. PER SQUARE FOOT.

Sheet lead	7
Sheet zinc	1·25 to 1·63
Sheet iron ($\frac{1}{16}$" thick)	3
„ corrugated	3·4
Slating	5 to 11
Tiles	7 „ 20
Pantiles	10
Cast-iron plates ($\frac{3}{8}$" thick)	15
Boarding ($\frac{3}{4}$" thick)	2·50
Thatch	6·50
Slates and iron laths	10
Sheet iron (16 W.G.) and laths	5
Corrugated iron and laths	5·5
Boarding and sheet iron (20 W.G)	6·5
Timbering of tiled and slated roofs, additional	5·5 to 6·5

WEIGHT OF ROOF FRAMING.

	Nature of Roof.	Clear Span.	Distance apart of Principal.	Weight per square foot of covered area in lb.		
				Purlins, &c.	Principal.	Total Ironwork.
		feet.	feet.			
Trussed.	Pent	15	..	..	..	3·5
Trussed.	Common Truss	37	5	1·1	3·5	4·6
Trussed.	"	40	12	2·0	3·5	5·5
Trussed.	"	50	10	..	..	3·0
Trussed.	"	54	14	6·5	3·0	9·5
Trussed.	"	55	6·5	4·6	7·0	11·6
Trussed.	"	72	20	4·2	2·8	7·0
Trussed.	"	84	9	2·6	5·9	8·5
Trussed.	"	100	14	..	..	7·0
Trussed.	"	130	26	0·8	5·6	6·4
Bowstring.	Manchester	50	11	..		9·6
Bowstring.	Lime Street	154	26	..	4·9	..
Bowstring.	Birmingham	211	24	..	7·3	11·0
Arched.	Strasburg Railway	97	13	..	..	12·0
Arched.	Paris Exchange	153	26	9·5	5·5	15·0
Arched.	Dublin	41	16	3·4	7·3	10·7
Arched.	Derby	81·5	24	10·8	6·0	16·8
Arched.	Sydenham	120	..	7·9	3·9	11·8
Arched.	"	72	..	8·4	2·9	11·3
Arched.	St. Pancras	240	29·33	7·4	17·1	24·5
Arched.	Cremorne	45	14·5	6·2	5 3	11·5

From 'Wrought-iron Bridges and Roofs.'—Unwin.

ROAD BRIDGES.

WEIGHT OF PLATFORM, &c.

Planking and joists	30 lb. per square foot.
Broken stone, or gravel, roadway	100 " "
Densely packed crowd	120 " "
Heavy draught horse	1400 lb.

RAILWAY BRIDGES.

WEIGHT OF PLATFORM, &c.

Rails	·03 tons per foot run of *each*
Ballast	·15 to ·21 line of rail.
Timbering	·07 to ·17.
Platform girder	·10 to ·25.

WEIGHTS OF LOCOMOTIVES AND TENDERS, G.N.R.

Tank Engine.

	Tender.		Engine.
Weights	13 tons 10 cwt.	14 tons 10 cwt.	12 tons 0 cwt.
Distances	12 ft. 9 in. —	— 7 ft. 6 in.	

Express Passenger Engine and Tender.

	Tender.			Engine.			
Weights	10 *t.* 8 *c.*	9 *t.* 18 *c.*	7 *t.* 16 *c.* ‖	8 *t.* 18 *c.*	15 *t.* 3 *c.*	7 *t.* 17 *c.*	7 *t.* 7 *c.*
Distances	6′ 6″ —	6′ 6″ —	7′ 1″ —	8′ 8″ —	7′ 9″ —	6′ 6″	

Express Passenger Engine (four coupled) and Tender.

	Tender.			Engine.		
Weights	10 *t.* 8 *c.*	9 *t.* 18 *c.*	7 *t.* 16 *c.* ‖	12 *t.* 0 *c.*	14 *t.* 10 *c.*	9 *t.* 12 *c.*
Distances	6′ 6″ —	6′ 6″ —	8′ 6″ —	8′ 3″ —	9′ 6″	

Goods Engine (six coupled) and Tender.

	Tender.			Engine.		
Weights	10 *t.* 8 *c.*	9 *t.* 18 *c.*	7 *t.* 16 *c.* ‖	11 *t.* 10 *c.*	14 *t.* 0 *c.*	9 *t.* 2 *c.*
Distances	6′ 6″ —	6′ 6″ —	7′ 9″ —	8′ 3″ —	7′ 3″	

WORKING STRENGTH OF VARIOUS MATERIALS.

Materials.	Safe load in lb. per square inch.			Modulus of Elasticity E.
	Tension.	Pressure.	Shearing.	
Wrought-iron Bars	10,400	10,400	7,800	28,500,000
Wrought-iron Plates	10,000	10,000	7,500	
Drawn Iron Wire	13,200	..	..	
Cast Iron	3,600	10,000	2,700	17,000,000
Soft Steel, unhardened	17,700	17,700	13,200	
„ hardened and tempered	35,400	35,400	26,600	30,000,000
Steel Wire	27,300	..	..	
Phosphor Bronze	9,900	..	7,400	14,000,000
Wood Ash α*	1,700	940	..	
Wood Ash β	.	510	..	
„ Oak α	1,560	940	100	1,500,000
„ Oak β	..	510	..	
„ Beech α	1,700	940	85	
„ Beech β	..	510	..	1,400,000
„ Pine α	1,000	620	55	
„ Pine β	..	310	..	
Good Brickwork	..	140	..	
Ordinary „	..	85	..	
Stone	..	200	..	

* α, stress parallel to fibres.
β, stress perpendicular to fibres.

NOTE.—*The above values of the safe load may be taken for structures subject to travelling load. For structures subject to dead load only, these values may, in the case of Iron and Steel, be multiplied by* $\frac{3}{2}$.

LONDON: PRINTED BY WILLIAM CLOWES AND SONS, LIMITED, STAMFORD STREET AND CHARING CROSS.

Fig. 39.

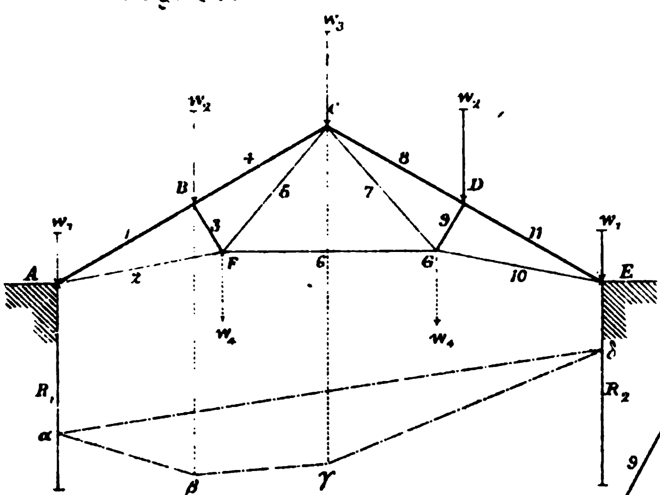


Fig. 39a

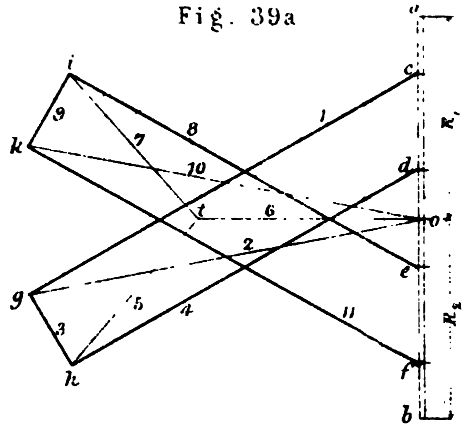


Fig. 39b.

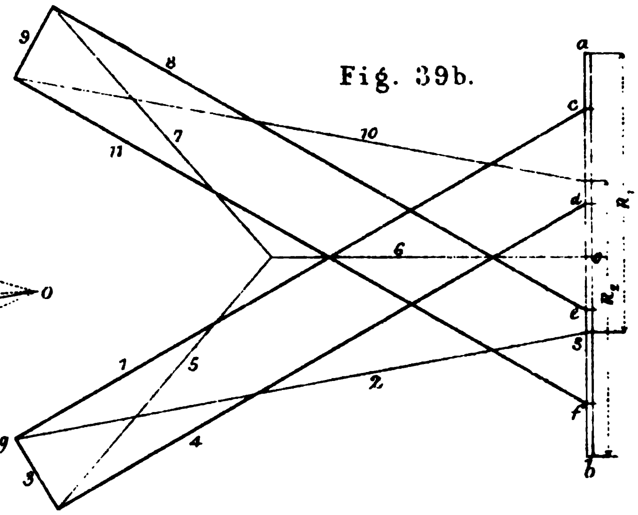


Fig. 39c.

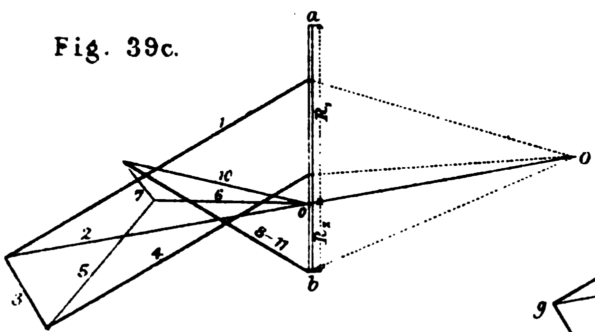


Fig. 40.

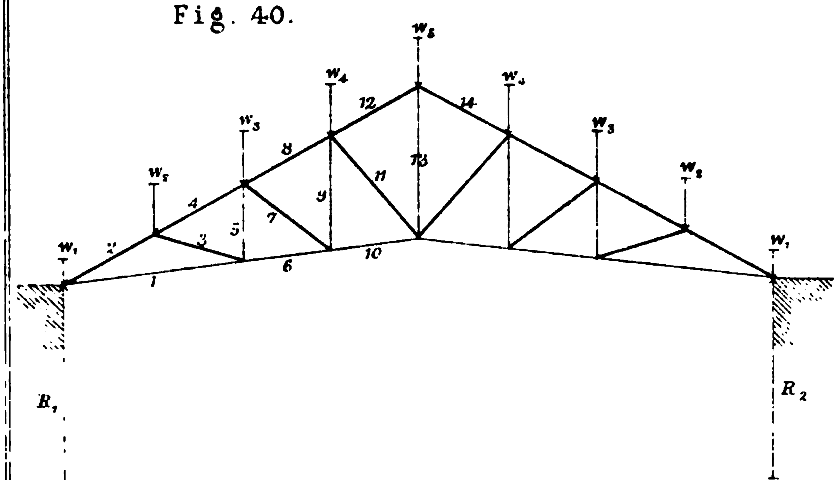
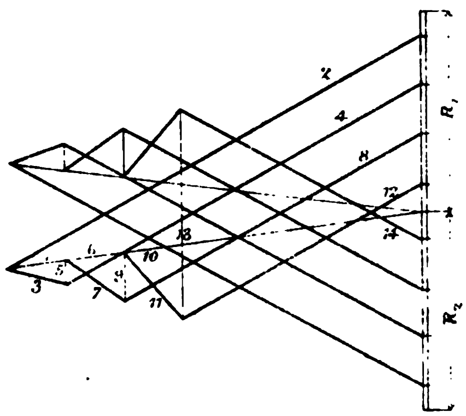


Fig. 40a.



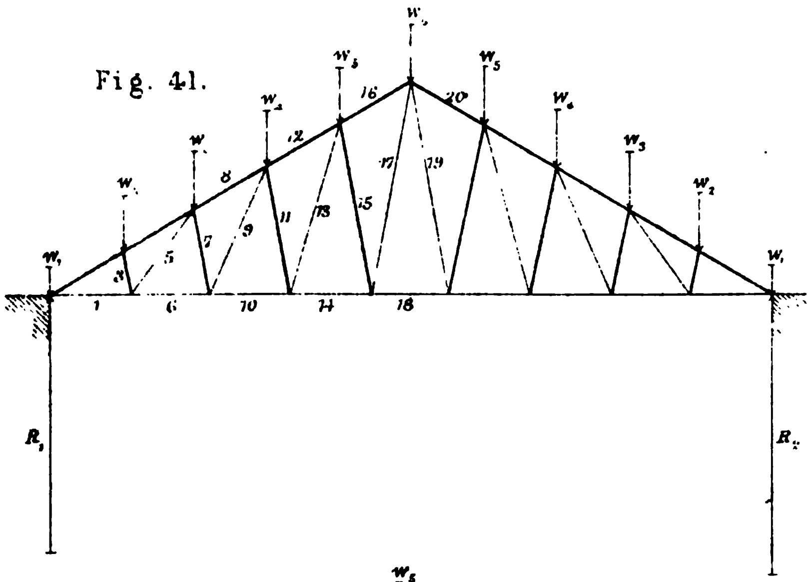


Fig. 41.

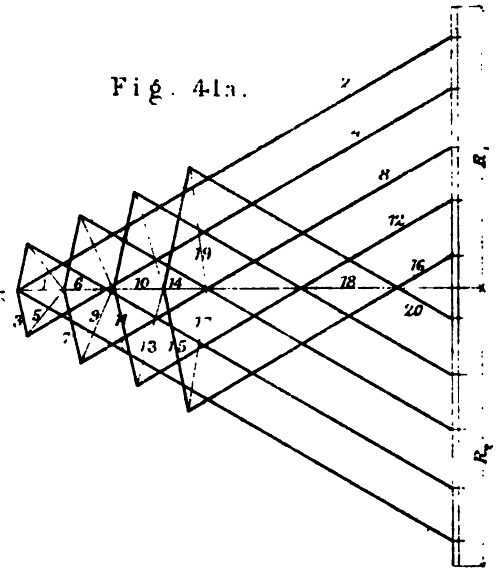


Fig. 41a.

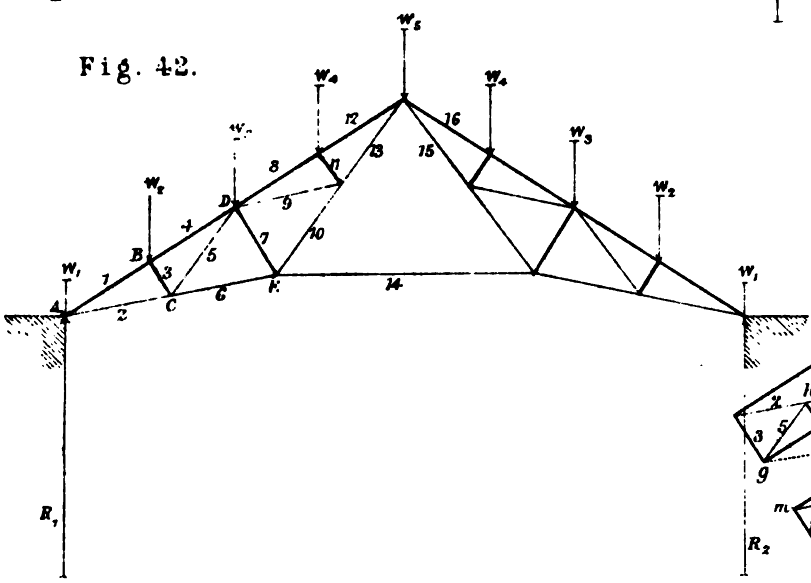


Fig. 42.

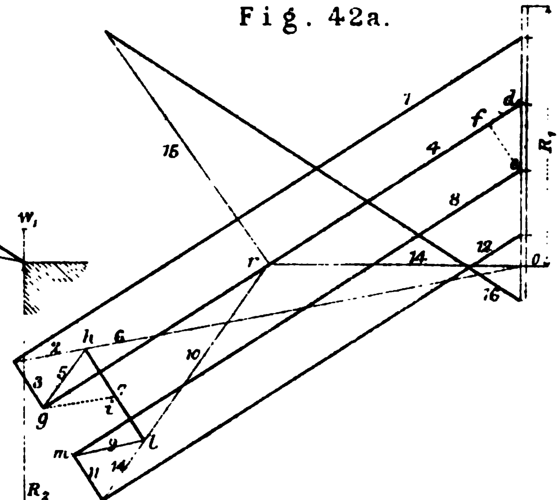


Fig. 42a.

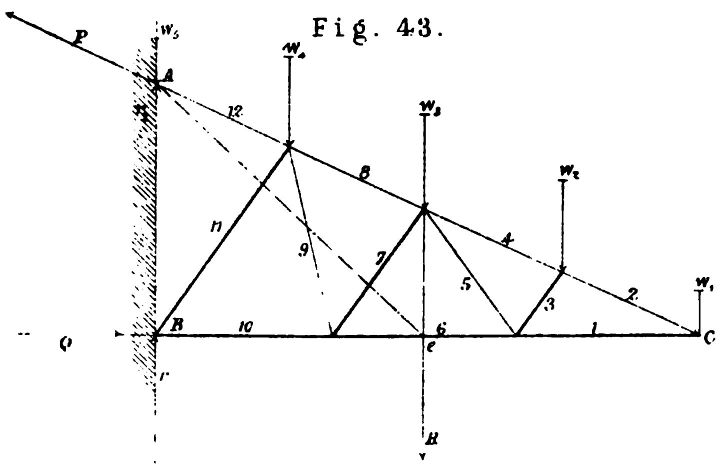


Fig. 43.

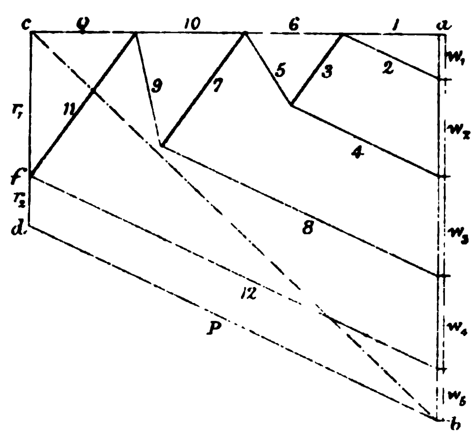


Fig. 43a.

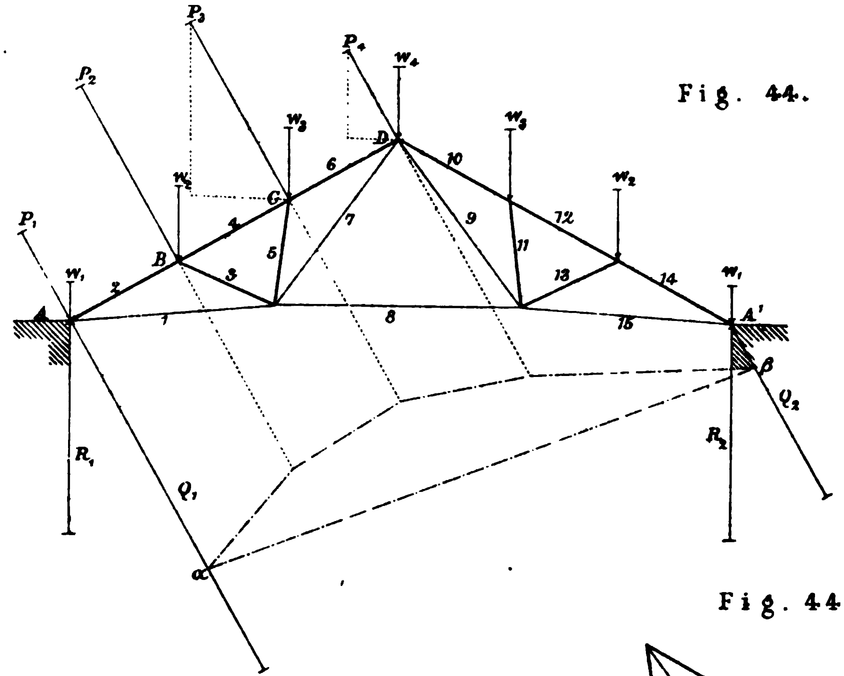


Fig. 44.

Fig. 44a.

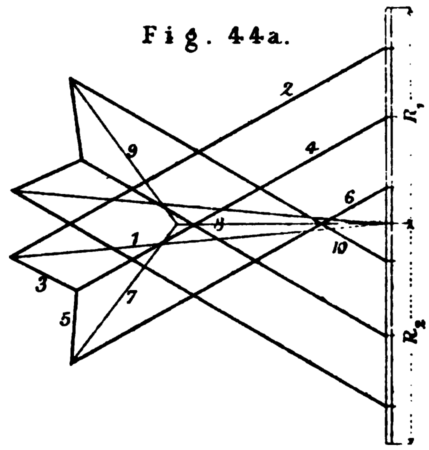


Fig. 44b.

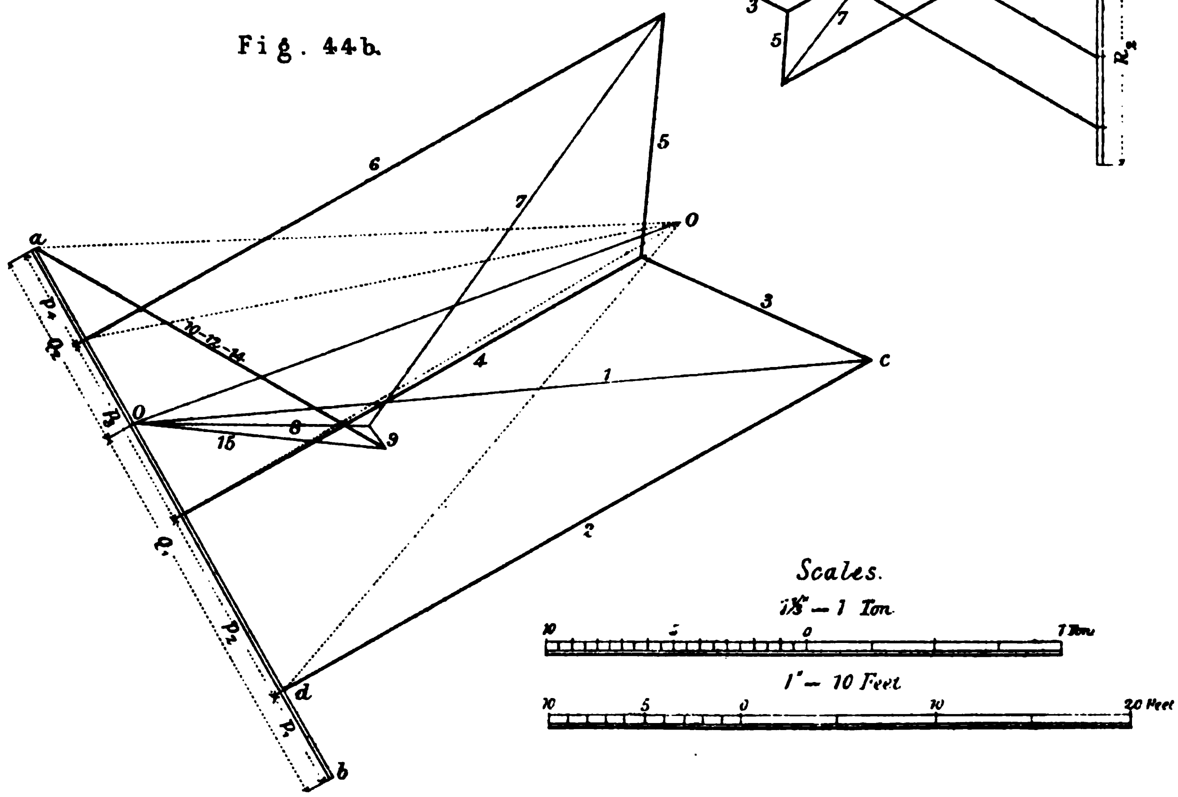


Fig. 44c.

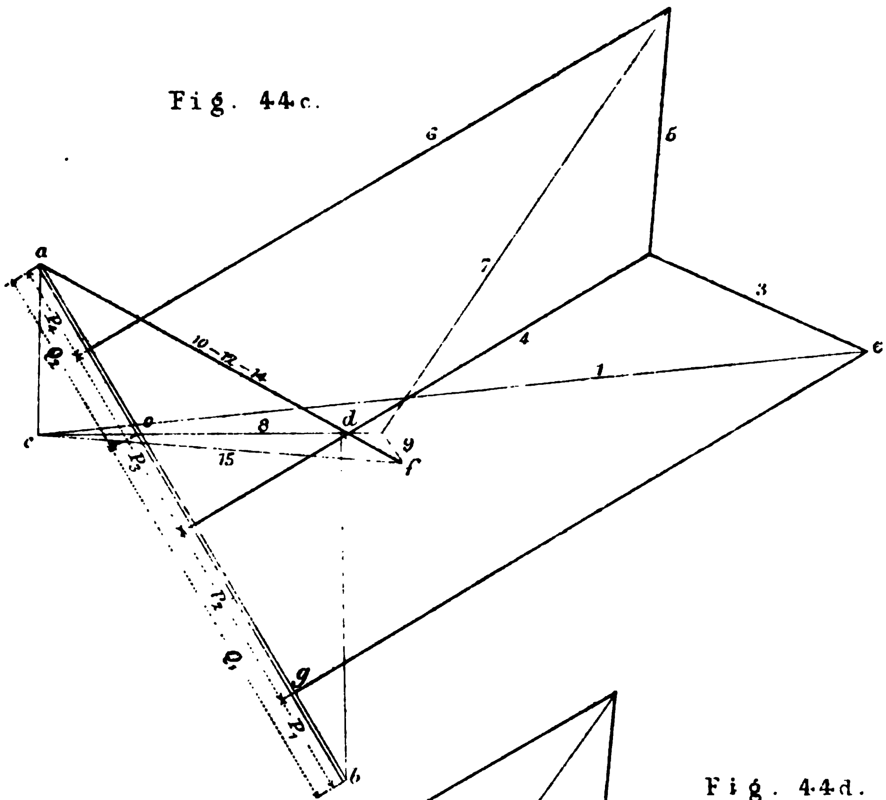
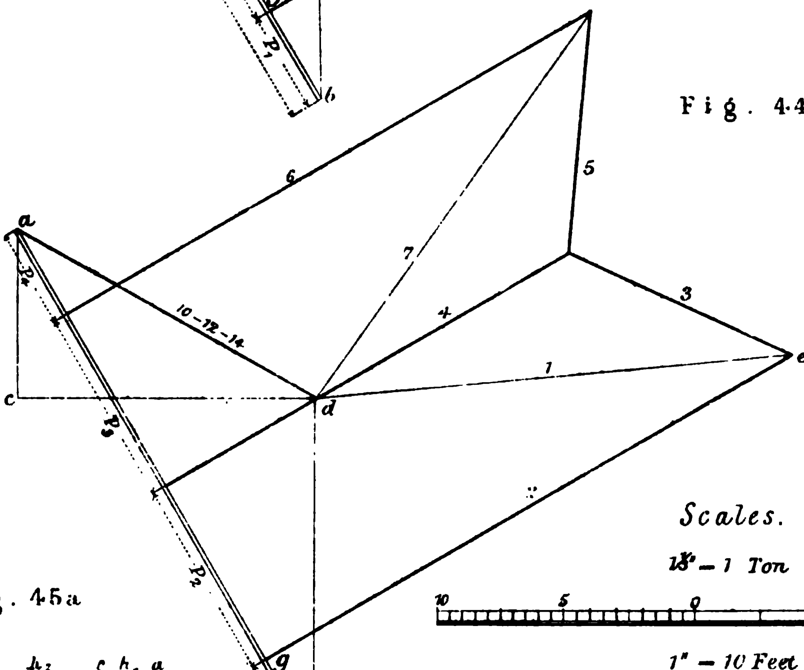


Fig. 44d.



Scales.

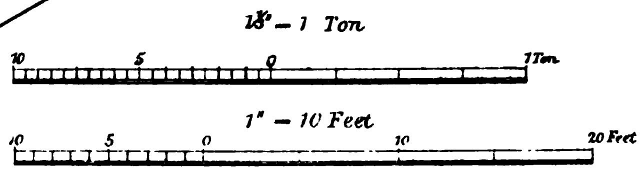


Fig. 45a

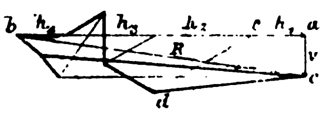


Fig. 45.

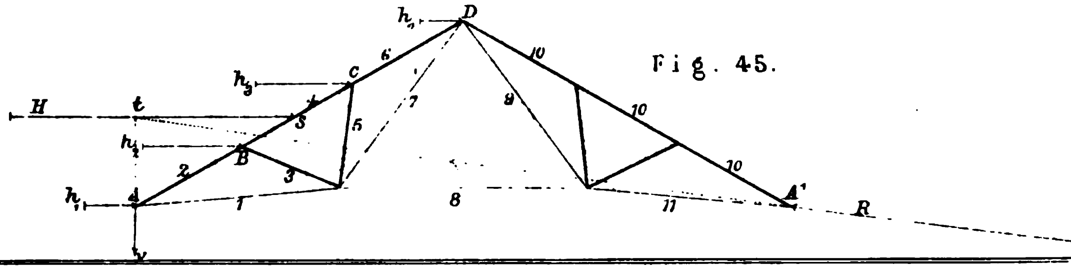


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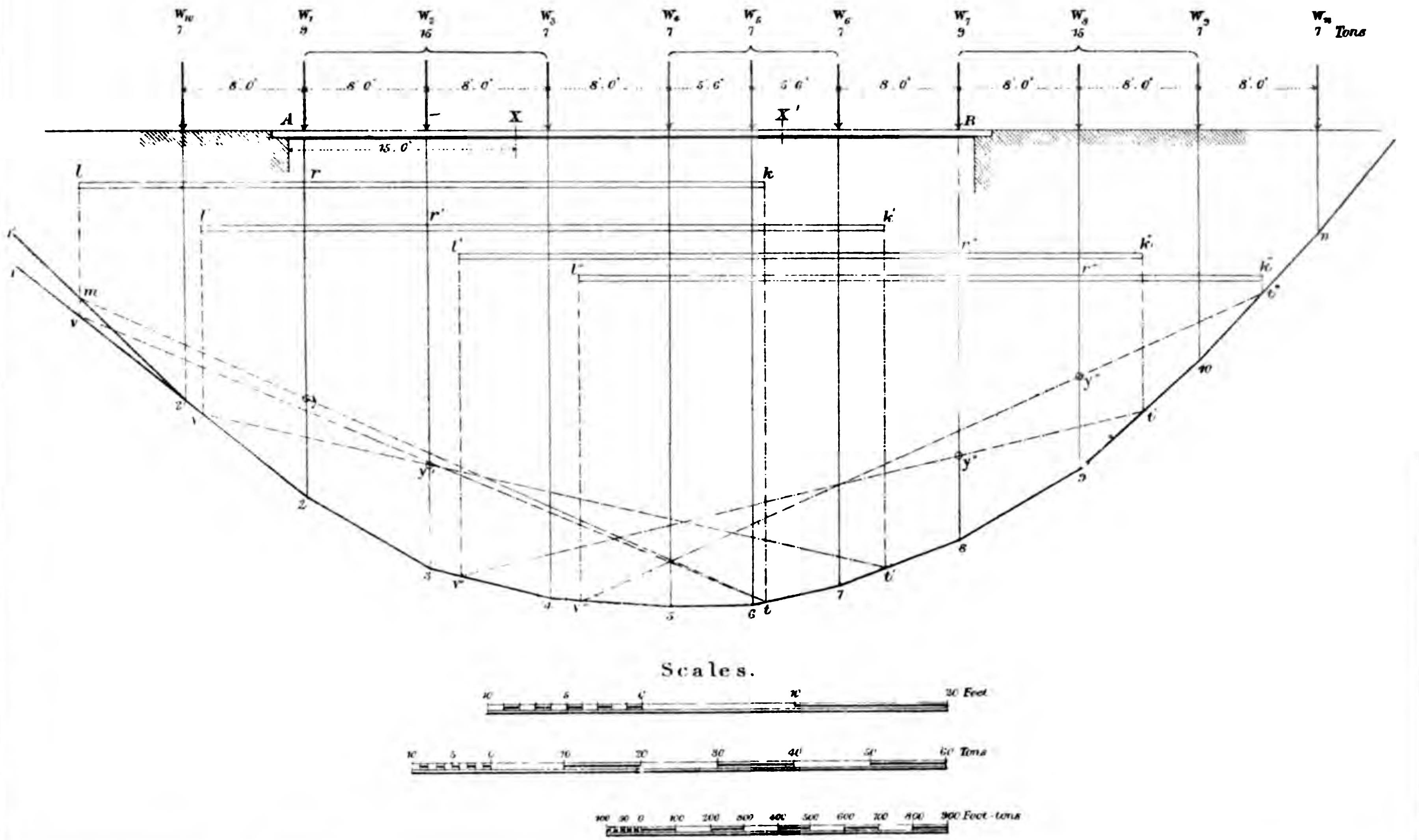
Fig. 67.


W10 7
W1 9
W2 15
W3 7
W4 7
W5 7
W6 7
W7 9
W8 15
W9 7
W11 7 Tons
8' 0"
5' 6"
A
B
X
X'
15' 0"
Scales.
30 Feet
60 Tons
900 Foot-tons

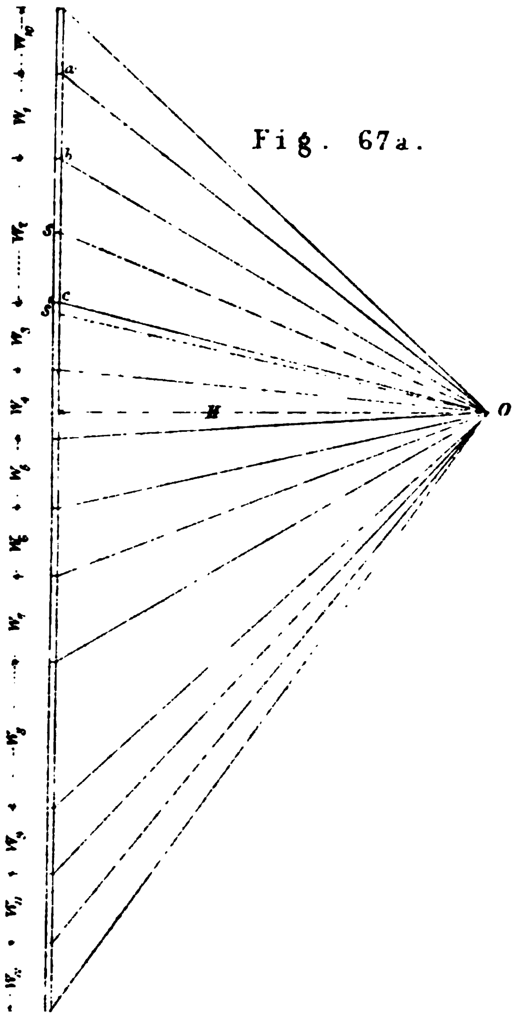


Fig. 67a.

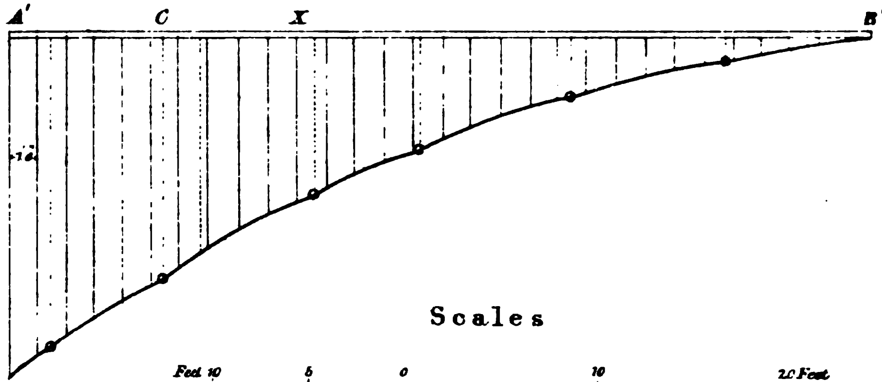
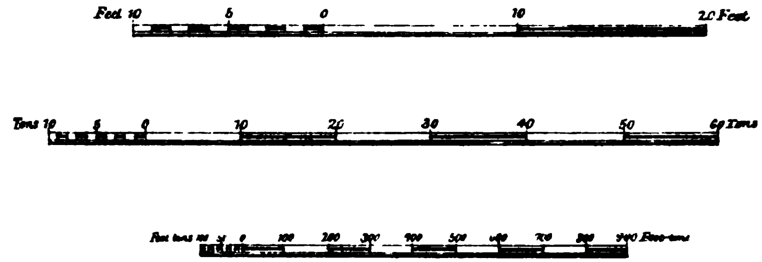


Fig. 67b.

Scales



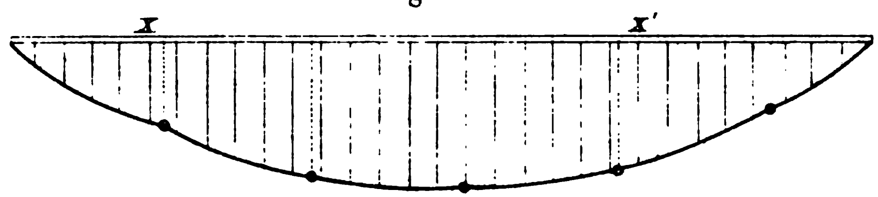
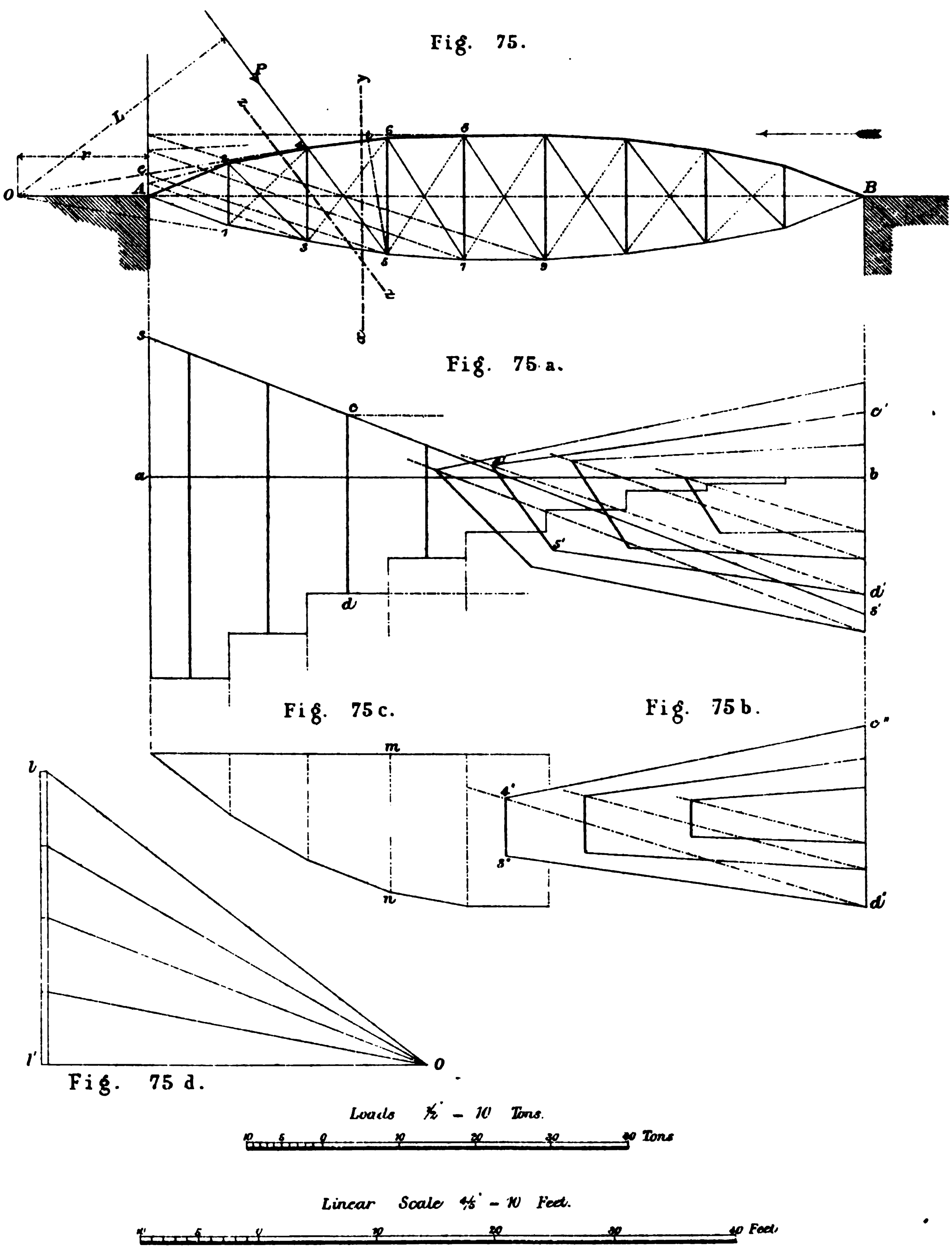


Fig. 67c.



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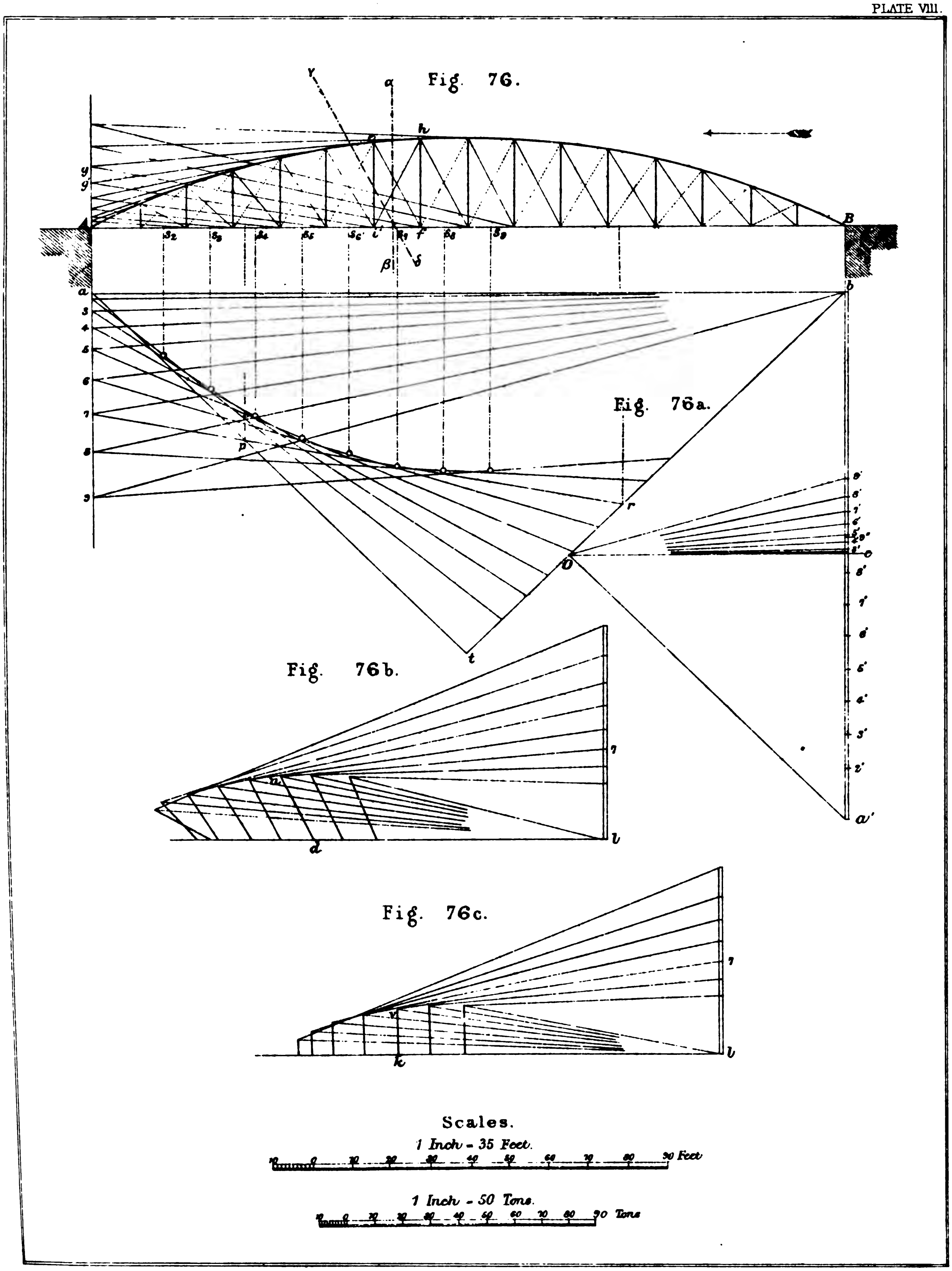

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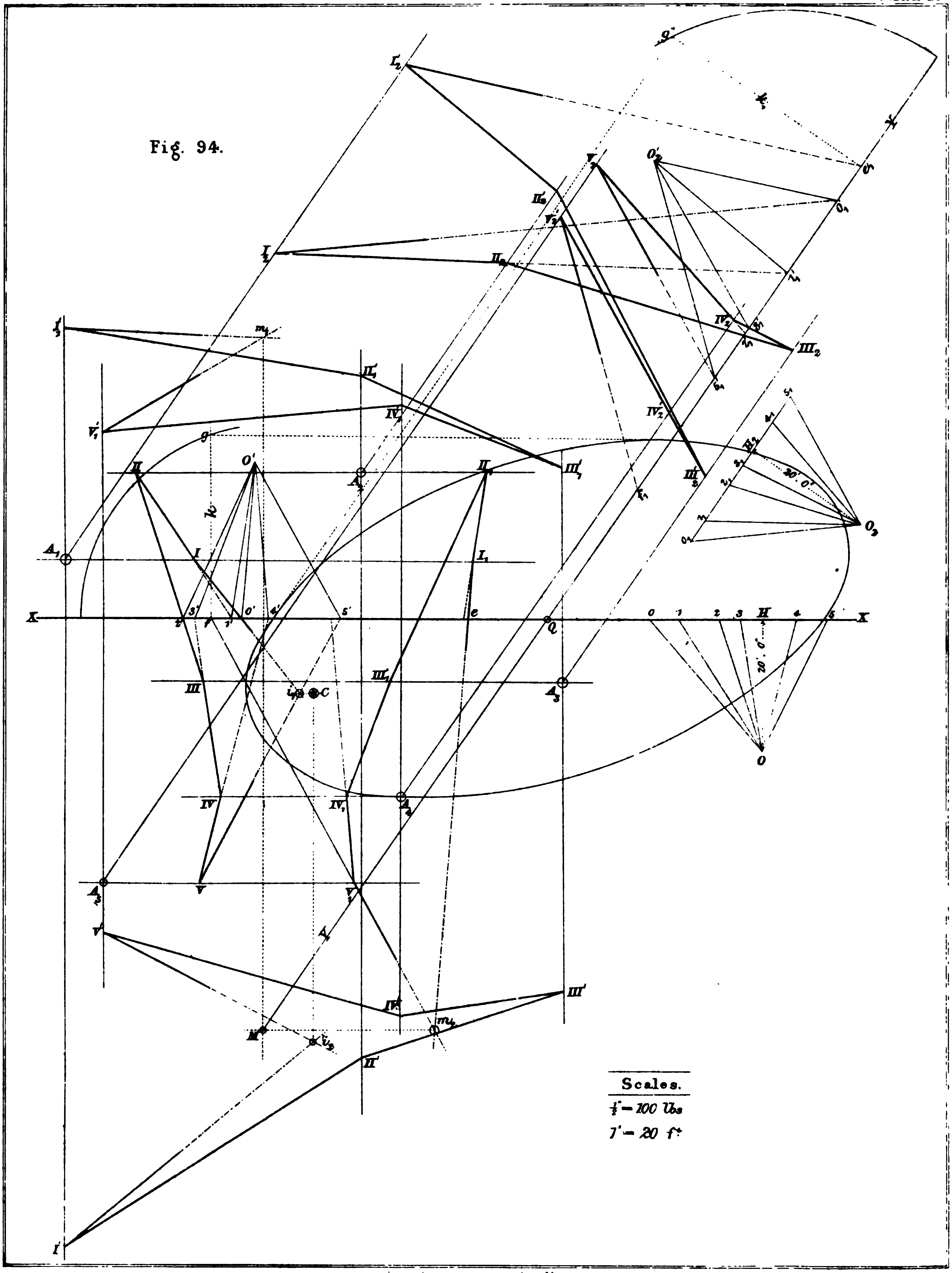

Fig. 94.

Fig. III.

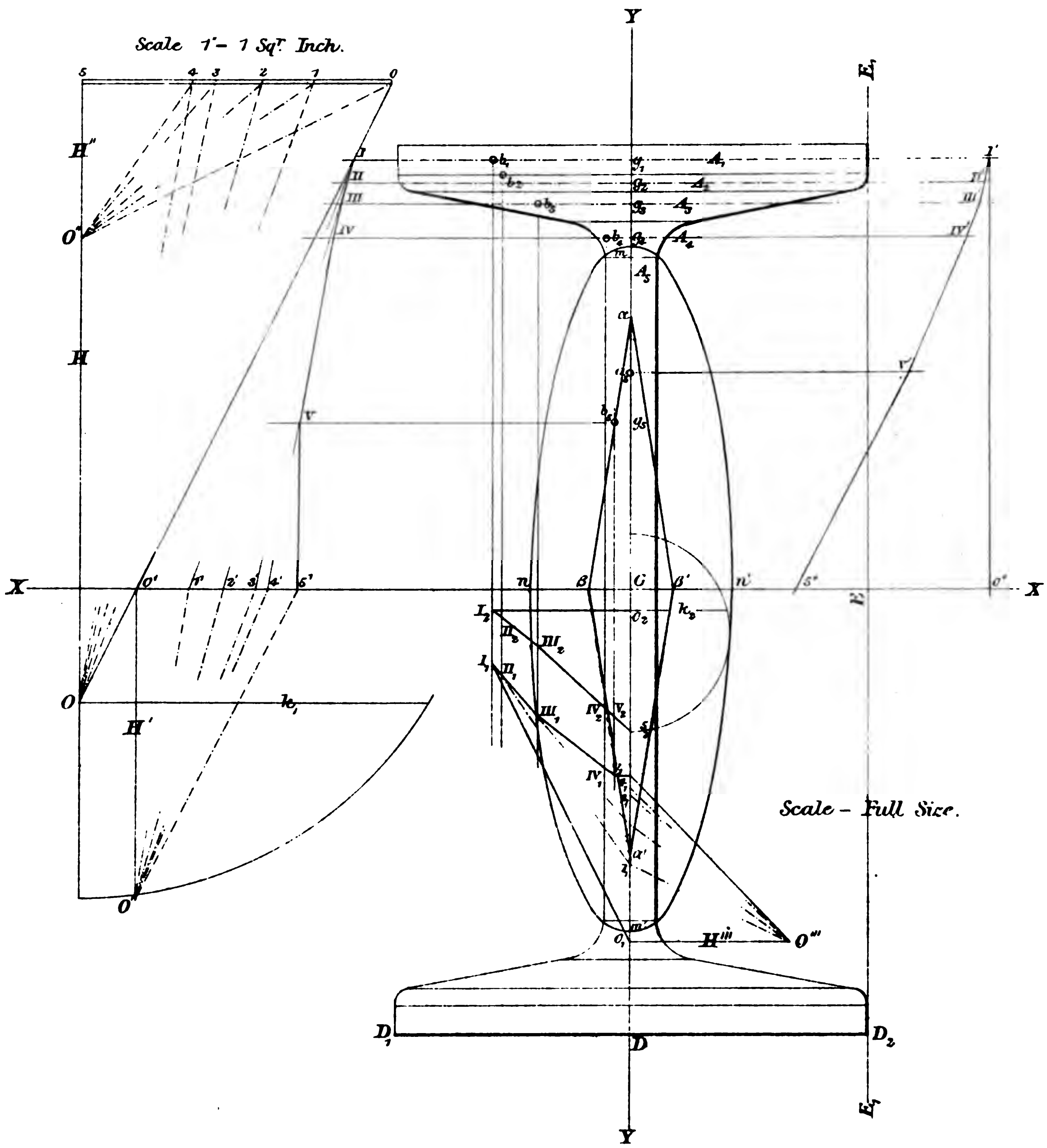

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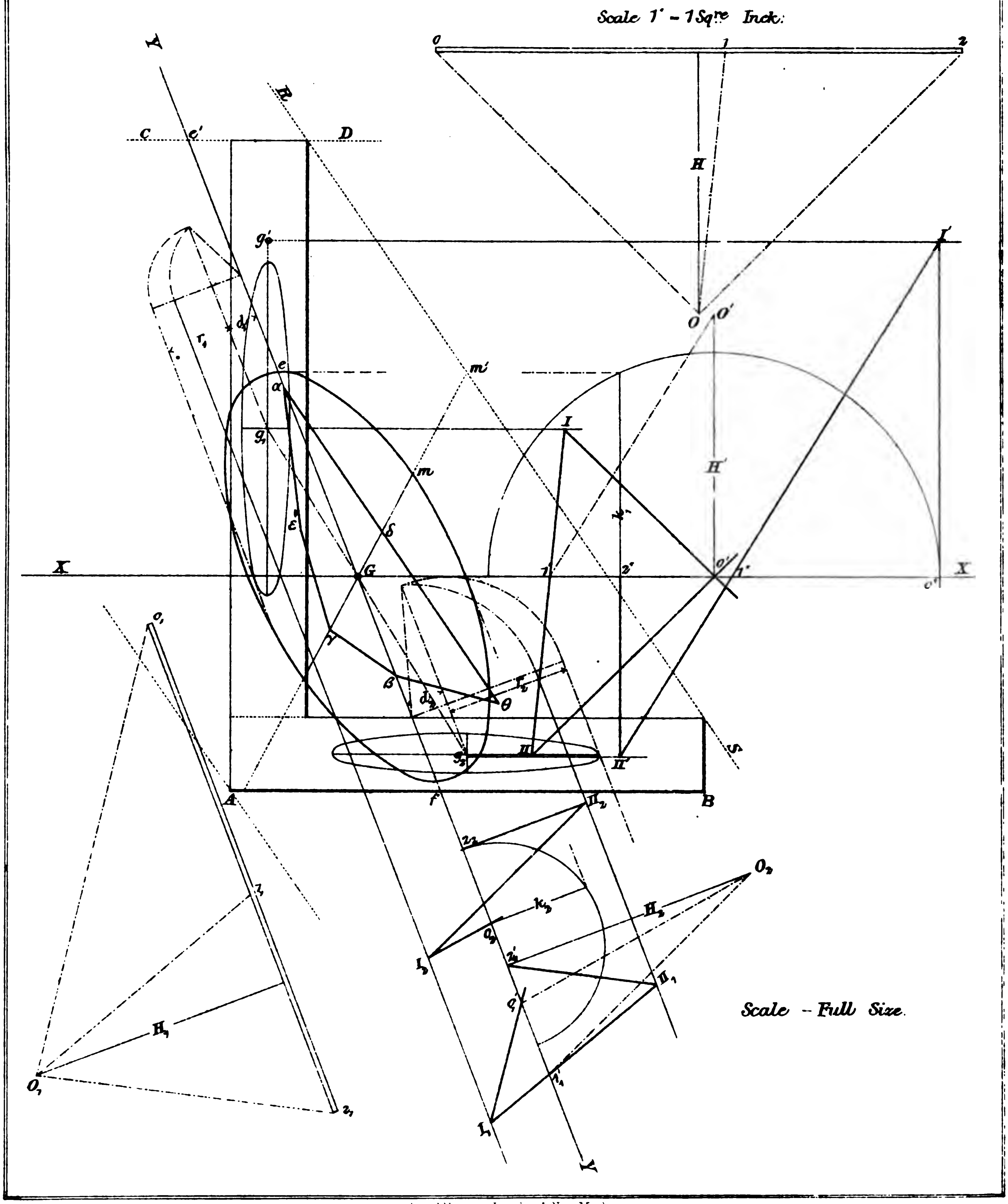
Fig. 112.



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